

# VORTEX LIQUIDS AND THE GINZBURG-LANDAU EQUATION

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**ABSTRACT.** We establish vortex dynamics for the time-dependent Ginzburg-Landau equation for asymptotically large numbers of vortices for the problem without a gauge field and either Dirichlet or Neumann boundary conditions. As our main tool, we establish quantitative bounds on several fundamental quantities, including the kinetic energy, that lead to explicit convergence rates. For dilute vortex liquids we prove that sequences of solutions converge to the hydrodynamic limit.

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## 1. INTRODUCTION

Let  $u : \Omega \rightarrow \mathbb{C}$  satisfy the scaled Ginzburg-Landau equation

$$(1.1) \quad \frac{1}{|\log \varepsilon|} \partial_t u = \Delta u + \frac{1}{\varepsilon^2} u (1 - |u|^2)$$

with either Dirichlet boundary conditions

$$(1.2) \quad u = e^{in\theta + i\varphi_\star} \text{ on } \partial\Omega$$

with  $\varphi_\star \in C^2$ ,  $\int_{\partial\Omega} \partial_\tau \varphi_\star = 0$ , so  $\deg(u; \partial\Omega) = n$ , or Neumann boundary conditions

$$(1.3) \quad \partial_\nu u = 0 \text{ on } \partial\Omega.$$

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We take  $\Omega$  to be a smooth, simply connected domain containing the origin. Equation (1.1) models the dynamic behavior of superconductors when the electromagnetic field potential is absent. When a gauge field is present, the corresponding Gorkov-Eliashberg equations

$$(1.4) \quad \begin{aligned} \partial_\Phi u &= \nabla_A^2 u + \frac{1}{\varepsilon^2} u (1 - |u|^2) \\ E &= -\operatorname{curl} \operatorname{curl} A + j_A(u), \end{aligned}$$

where  $\partial_\Phi = \partial_t + i\Phi$ ,  $E = \partial_t A + \nabla\Phi$ , and  $j_A(u) = (iu, \nabla_A u)$ , provide a more complete model of superconductivity.

In order to describe the behavior of solutions of (1.1) with small  $\varepsilon$  we define some fundamental quantities including:

$$(1.5) \quad \text{energy density} \quad e_\varepsilon(u) = \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2$$

$$(1.6) \quad \text{supercurrent} \quad j(u) \equiv (iu, \nabla u)$$

$$(1.7) \quad \text{vorticity/Jacobian} \quad J(u) \equiv \det \nabla u = \frac{1}{2} \operatorname{curl} j(u).$$

Here  $(\cdot, \cdot)$  denotes the real scalar product of two complex numbers, so  $(a, b) = \frac{1}{2}(\bar{a}b + a\bar{b})$  for  $a, b \in \mathbb{C}$ . Solutions to equation (1.1) diffuse the Ginzburg-Landau energy

$$(1.8) \quad E_\varepsilon(u) = \int_\Omega e_\varepsilon(u)$$

via the identity

$$(1.9) \quad E_\varepsilon(u(t)) + \int_0^t \int_\Omega \frac{|\partial_t u|^2}{|\log \varepsilon|} = E_\varepsilon(u(0)).$$

**1.1. Vortex dynamics and vortex liquids.** A prominent feature of type II superconductivity is the presence of localized regions, called *vortices*, where superconductivity vanishes. In particular there exist some points  $\{a_j\}_{j=1}^n$  in  $\Omega$  where  $|u(a_j, \cdot)| \approx 0$ . Furthermore, about each vortex the winding number of the phase is *quantized*; in particular

$$\frac{1}{2\pi} \int_{\partial B_r(a_j)} \tau \cdot j(u) \approx d \in \mathbb{Z} \setminus \{0\}.$$

In the vicinity of each vortex the Ginzburg-Landau energy  $E_\varepsilon(u)$  blows up at the rate  $\pi |\log \varepsilon| + O(1)$ . Bethuel-Brezis-Helein showed in [3] that minimizers of the Ginzburg-Landau energy (1.8) can be expanded further up to second order

$$E_\varepsilon(u) = n(\pi |\log \varepsilon| + \gamma_0) + W(a) + O(1),$$

where  $\gamma_0$  is a universal constant and

$$W(a) = -\pi \sum_{i \neq j} \log |a_i - a_j| + \text{boundary effects}$$

is a *renormalized energy* and the winding number about each vortex is one. This renormalized energy is precisely the bounded domain version of the *Kirchhoff-Onsager* functional that arises in two dimensional incompressible Euler equations and other settings. The renormalized energy will be discussed in more detail in Section 2 and

Section 9. From back-of-the-envelope calculations one finds that  $J(u)$  is quantized and looks like a sum of integer-weighted delta functions; and so, for small  $\varepsilon$  one finds that

$$J(u) \approx \frac{e_\varepsilon(u)}{|\log \varepsilon|} \approx \pi \sum_{j=1}^n \delta_{a_j}$$

in case when the winding number about each vortex equals one, and as  $\varepsilon \rightarrow 0$ ,  $u$  limits to

$$(1.10) \quad u_\star = \prod_{j=1}^n \frac{x - a_j}{|x - a_j|} e^{i\psi_\star}$$

where  $\psi_\star$  is  $H^1(\Omega)$ . This  $u_\star$  is referred to as the *canonical harmonic map* when  $\psi_\star$  is a harmonic function. This limiting behavior was established in many situations, see for example [3, 32, 42, 19, 20].

When dynamics (1.1) are turned on, these vortices move according to the gradient flow of the Kirchhoff-Onsager energy:

$$(1.11) \quad \dot{a}_j = -\frac{1}{\pi} \nabla_{a_j} W.$$

The  $|\log \varepsilon|$  factor in front of (1.1) is the critical time scale on which vortices will move and can be thought of as the length of time it takes the unscaled time dependent Ginzburg-Landau equation to move an  $O(|\log \varepsilon|)$  amount of energy an  $O(1)$  distance. That vortices satisfy (1.11) in the limit was the subject of a formal asymptotic study by E [12]. Later, arguments of Lin [30] and Jerrard-Soner [21] provided rigorous justification of the limit. Both [30] and [21] assume that the number of vortices is uniformly bounded as  $\varepsilon \rightarrow 0$ . The limit equation (1.11) is the gradient flow of  $W$  just as (1.1) is the (rescaled) gradient flow of the integrated energy density  $\int_\Omega e_\varepsilon(u)$ . The similarity in structure can also be seen by the energy dissipation identity

$$(1.12) \quad W(a(t)) + \int_0^t |\dot{a}(s)|^2 = W(a(0)).$$

This structure was exploited to give a more abstract proof of the motion law by Sandier-Serfaty [43] in their  $\Gamma$ -convergence of gradient flows framework.

In recent years there have been significant advances in understanding the dynamics of a finite numbers of vortices by Bethuel-Orlandi-Smets [4] on  $\mathbb{R}^2$  and by Serfaty [48] on bounded domains. These results allow for much weaker initial conditions, handle collisions of plus/minus vortices, and describe the dynamical behavior of higher degree vortices.

On the other hand, the behavior of the time dependent Ginzburg-Landau equations with asymptotically large numbers of vortices has received mostly formal treatment. The question of how large numbers of vortices behave in superconductors is important from both experimental and numerical perspectives. In the former, typical superconductors contain many millions of vortices per sample [6, 14] so the large vortex problem is a fundamental feature of high  $T_C$  superconducting devices. In the latter, point vortex methods provide a useful class of numerical algorithms for simulating challenging PDE's, like vortex sheets; hence, (1.11) is a reasonable numerical approximation of the limiting mean field equation with vortex sheet initial data.

In [13] E looks at how the planar analogue of (1.11) behaves in a mean field sense as  $n \rightarrow \infty$ . Defining the vortex density function  $\omega_n = \frac{1}{n} \sum_{j=1}^n \delta_{a_j(t)}$ , the author shows that the limiting density,  $\omega = \lim_{n \rightarrow \infty} \omega_n$ , formally satisfies a weak PDE of the form

$$\begin{aligned} \partial_t \omega + \operatorname{div}(\omega v) &= 0 \\ v &= \nabla (\Delta^{-1}) \omega \end{aligned}$$

after rescaling time  $t$ . Subsequently, this ODE limit on  $\mathbb{R}^2$  was rigorously established by Lin-Zhang [34]. Furthermore, these authors show that this Euler-type equation has many regularizing features that will be discussed later.

There are many similarities between this ODE limit problem and ODE limit problem arising from the point vortex method for the Euler equations. In the latter case it was shown by Schochet [46] and Liu-Xin [36] that the vortex density function for Euler point vortices on  $\mathbb{R}^2$ , which follow the Kirchhoff law

$$\dot{a}_j = -\frac{1}{\pi} \nabla_{a_j}^\perp W(a),$$

limit to a weak Delort solution to the incompressible Euler equations on  $\mathbb{R}^2$ . Due to the similarities of the problem, Lin-Zhang [34] use the approach of [36] to prove the associated hydrodynamic limit of (1.1) on planar domains.

The present work is the first to directly couple the Ginzburg-Landau equation to a mean field PDE. All previous works either prove a PDE to ODE limit for a bounded number of vortices or an ODE to mean field PDE limit. Our quantitative results enable us to take the diagonal limit in a rigorous way.

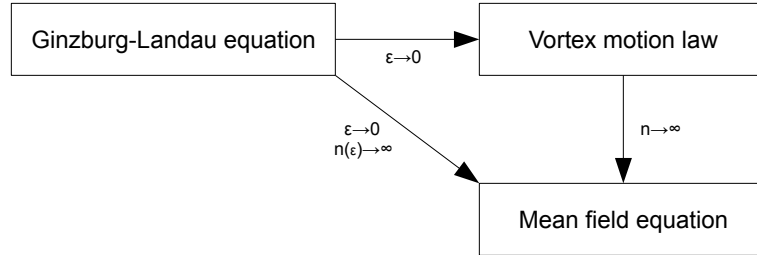


FIGURE 1. Limiting from the Ginzburg-Landau equation directly to the mean field equation

In order to make the direct connection between the Ginzburg-Landau equation and the limiting mean field equation, it is necessary to establish two steps. The first of which entails a proof that (1.1) can accept asymptotically large numbers of vortices for long-enough times. The second step involves coupling these Ginzburg-Landau solutions to an appropriate hydrodynamic limit of (1.11) on bounded domains.

**1.2. Results.** We now state our first theorem which establishes vortex dynamics for asymptotically large numbers of vortices. Though the initial conditions require well-preparedness and all degree plus-one vortices, such conditions can be softened a bit by combining with the work of [4, 48]. In the following we let

$$A \lesssim B \quad \text{if } A \leq CB$$

for some  $C$  that depends only on  $\Omega$ . Next we define the *excess energy*

$$D(a(t)) = E_\varepsilon(u(t)) - [n(\pi |\log \varepsilon| + \gamma_0) + W(a(t))],$$

which will be used to control the deviation of the vortex path from the path defined by the ODE (1.11). We also define

$$\rho_\star = \frac{1}{4} \min_{0 \leq t \leq T} \{ \min_{j \neq k} |a_j(t) - a_k(t)|, \min_j \text{dist}(a_j(t), \partial\Omega) \}$$

to be a measure of how close vortices can approach each other or the boundary on the set time scale. Finally, we introduce a weak topology related to the length of a minimal connection, see [5],

$$\|f\|_{\dot{W}^{-1,1}(\Omega)} = \sup_{\substack{\|\nabla \phi\|_{L^\infty(\Omega)} \leq 1 \\ \phi \in W_0^{1,\infty}(\Omega)}} \left| \int_\Omega \phi f \right|.$$

This norm provides a good scale-invariant measure of the distance of  $J(u)$  and  $\frac{e_\varepsilon(u)}{|\log \varepsilon|}$  to a sum of delta functions. In particular if  $|a_j - b_j| \leq \rho_\star$  for  $j = 1, \dots, n$  then

$$\left\| \sum_j \delta_{a_j} - \sum_j \delta_{b_j} \right\|_{\dot{W}^{-1,1}(\Omega)} = \sum_j |a_j - b_j|.$$

We can now state our first theorem which supplies a long time existence result for asymptotically large numbers of vortices in the dilute regime.

**Theorem 1.1.** *Suppose  $u$  solves (1.1) with either (1.2) or (1.3). Furthermore, let  $n \leq |\log \varepsilon|^{\frac{1}{200}}$  and  $\rho_\star \geq |\log \varepsilon|^{-\frac{1}{100}}$  and*

$$(1.13) \quad D(a(0)) \lesssim |\log \varepsilon|^{-\frac{2}{5}}$$

$$(1.14) \quad \|J(u(0)) - 2\pi \sum_{j=1}^n \delta_{a_j(0)}\|_{\dot{W}^{-1,1}(\Omega)} \leq C_\star |\log \varepsilon|^{-\frac{1}{3}}$$

then for all  $0 \leq t \leq T$  we have

$$\begin{aligned} \left\| \frac{e_\varepsilon(u)(t)}{|\log \varepsilon|} - \sum_{j=1}^n \pi \delta_{a_j(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} &\lesssim |\log \varepsilon|^{-\frac{1}{4}} \\ \|J(u)(t) - \sum_{j=1}^n \pi \delta_{a_j(t)}\|_{\dot{W}^{-1,1}(\Omega)} &\lesssim |\log \varepsilon|^{-\frac{1}{4}} \\ \int_{\Omega_{\rho_\star}(a)} e_\varepsilon(|u|) + \frac{1}{4} \left| \frac{j(u)}{|u|} - j(u_\star) \right|^2 &\lesssim |\log \varepsilon|^{-\frac{1}{5}} \\ \left\| \frac{j(u)}{|u|} - j(u_\star) \right\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim |\log \varepsilon|^{-\frac{1}{10}} \end{aligned}$$

where

$$(1.15) \quad T = \min \{ C \sqrt{\log |\log \varepsilon|} \frac{\rho_\star^2}{n}, \tau_0 \}$$

where  $\tau_0 = \inf_{0 \leq t \leq T} \{\rho_a(t) < \rho_\star\}$ ,  $a_j(t)$  solve (1.11), and  $C = C(\Omega)$ . Here  $\Omega_\rho(a) = \Omega \setminus \cup_{j=1}^n B_\rho(a_j)$ .

From Lemma 14 of [23] one can easily construct maps  $u_\star^\varepsilon(x; a)$  that satisfy the well-preparedness assumptions (1.13)–(1.14). In the case of a bounded number of vortices is well known that this hypothesis is not very important, since one can show that data will become well-prepared almost instantaneously due to strong convergence estimates, see [30, 21, 4, 48], and we have no reason to expect a different behavior here.

Given the result above, we can prove that the sequence of solutions converge in a prescribed sense to the expected hydrodynamic limit. In this theorem we study only Dirichlet boundary conditions (1.2) since we need to have the vortex motion law hold for times of order  $O(n^{-1})$ , and in the Neumann case (1.3) vortices will migrate to the boundary too quickly.

What type of equation do we formally expect the vortex density to satisfy in the limit? If we rescale time  $\bar{t} = nt$  then the limiting vortex density function  $\omega = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \delta_{a_j(t)}$  on a bounded domain should satisfy a weak form of the system

$$(1.16) \quad \begin{aligned} \partial_{\bar{t}} \omega + \operatorname{div}(\omega v) &= 0 \\ v &= \nabla (\Delta_{\mathcal{N}}^{-1}) \omega \end{aligned}$$

where  $\Delta_{\mathcal{N}}^{-1} : g \rightarrow w$  arises through the Poisson problem

$$(1.17) \quad \begin{aligned} \Delta w &= g \text{ in } \Omega \\ \partial_\nu w &= \partial_\tau \theta \text{ on } \partial\Omega \end{aligned}$$

for  $\theta = \arctan(y/x)$ . Note consistency requires  $\int_\Omega g = 2\pi$  due to the Neumann boundary condition. To motivate a notion of an interior weak solution of (1.16) we follow Lin-Zhang [34]. If  $\omega$  is a smooth solution to (1.16), we multiply by  $\chi \in C_0^\infty(\Omega)$  and integrate by parts. Then  $-\int \int \partial_t \chi \omega - \int \int \partial_{x_k} \chi v_k \partial_{x_j} v_j = 0$ , where we used  $\omega = \operatorname{div} v$  in the interior of  $\Omega$ . Finally, using  $\partial_1 v_2 - \partial_2 v_1 = 0$  and integrating by parts again yields (1.18) below. We note that this definition is similar to the one introduced in [9, 38] for weak solutions to the Euler equations except that the associated test functions are exchanged.

**Definition 1.2.** We say  $\omega$  is a generalized interior weak solution to (1.16) if for all  $\chi \in C_0^\infty(\Omega)$

$$(1.18) \quad -\int_0^t \int_\Omega \omega \partial_t \chi + \int_0^t \int_\Omega \left( \frac{\partial_{x_1}^2 \chi - \partial_{x_2}^2 \chi}{2} \right) (v_1^2 - v_2^2) + \int_0^t \int_\Omega 2 \partial_{x_1} \partial_{x_2} \chi v_1 v_2 = 0$$

where  $v_j(x) = \partial_{x_j} \Delta_{\mathcal{N}}^{-1} \omega = \int \partial_{x_j} N(x, y) \omega(y) dy$  where  $N(x, y)$  is the Neumann function defined via (1.17).

We can now state our main result which shows that we can solve (1.16)–(1.17) with vortex sheet initial data via a sequence of point vortex methods with appropriate data.

**Theorem 1.3.** Assume that  $\omega_0 \in \mathcal{M} \cap \dot{H}^{-1}(\Omega)$  satisfies  $\omega_0 \geq 0$ ,  $\int_\Omega \omega_0 = 2\pi$ , and  $\operatorname{supp}(\omega_0) \subset \{\operatorname{dist}(x, \partial\Omega) \geq C_0 > 0\}$  for some constant  $C_0$ . Then there exists a sequence

of initial data  $u_{\varepsilon_n}(0)$  with  $n = |\log |\log \varepsilon_n||^{\frac{1}{4}}$  number of vortices that satisfies the hypotheses for Theorem 1.1 such that  $\frac{1}{n} \frac{e_{\varepsilon_n}(u_{\varepsilon_n}(0))}{\pi |\log \varepsilon_n|} \rightarrow \omega_0$  in  $\mathcal{M} \cap \dot{H}^{-1}$  as  $\varepsilon_n \rightarrow 0$ . Such initial data generates a sequence of solutions  $u_{\varepsilon_n}(t)$  of (1.1) with boundary condition (1.2) for times up to  $T = |\log |\log \varepsilon_n||^{\frac{1}{7}}$ .

Setting  $\bar{t} = nt$  and letting  $\omega_{\varepsilon}(\bar{t}) = \frac{1}{n} \frac{e_{\varepsilon}(u_{\varepsilon}(\bar{t}))}{\pi |\log \varepsilon|}$  then

$$\omega_{\varepsilon}(\bar{t}) \rightarrow \omega(\bar{t}) \text{ in } \mathcal{M} \cap \dot{H}^{-1}$$

where  $\omega(\bar{t})$  is a generalized interior weak solution, defined above, to

$$(1.19) \quad \begin{aligned} \partial_{\bar{t}} \omega + \operatorname{div}(v\omega) &= 0 \\ v &= \nabla (\Delta_{\mathcal{N}}^{-1}) \omega \end{aligned}$$

for all  $t \in [0, \infty)$ . Finally,  $v(t) \in L_{loc}^2(\Omega)$  for all time. Here  $\Delta_{\mathcal{N}}^{-1} f = w$  if

$$(1.20) \quad \begin{aligned} \Delta w &= f \text{ in } \Omega \\ \partial_{\nu} w &= \partial_{\tau} \theta \text{ on } \partial\Omega \end{aligned}$$

where  $\theta = \arctan(\frac{y}{x})$  and  $\int_{\Omega} f = 2\pi$ .

**1.3. Discussion.** The issue of whether the weak solution satisfies the correct boundary condition is a deep and difficult question. Since vorticity can (and should) concentrate on the boundary, it is difficult to acquire the necessary regularity to ensure the boundary conditions are achieved in the classical weak sense. Some recent progress has been made in [17] by establishing *boundary-coupled weak solutions* of the two dimensional incompressible Euler equations in exterior domains.

To make a fully consistent limit it would be interesting to study the question of uniqueness of the limiting mean field equation (1.19). In [34] the authors establish uniqueness for initial data in  $L^{\infty}$  with compact support for the planar problem. A similar study of regular solutions would be natural for (1.19)-(1.20) too.

From formal considerations of (1.19) the vortex density function satisfies  $\partial_t \omega + v \cdot \nabla \omega = -\omega^2$ , so along the trajectory of the induced velocity one sees that the density function should decay like  $t^{-1}$ . In fact for smooth initial data on  $\mathbb{R}^2$  Lin-Zhang [34] proved this fact, which implies that the vorticity spreads out quickly from a compact set. This behavior implies that we expect most vortices to be pushed to out the boundary in a similar fashion. This conforms to the picture presented in Sandier-Soret [45] for global minimizers of the functional  $E_{\varepsilon}(u)$  on bounded domains constrained to the boundary condition of the type  $u = e^{in\theta}$  and  $n \rightarrow \infty$ . Sandier-Soret show that vortices accumulate close to the boundary of the domain as  $n$  grows asymptotically large. Taken together, we should view Theorem 1.3 as a mean field description of the vortex density for times in the *mesoscale* in the interior of the domain.

The dilute density of the vortex liquid results from two issues. The first is that we use energy comparison and a Gronwall inequality to pin the vortex positions to the ODE (1.11). This results in an upper bound  $T \lesssim \frac{\rho_x^2}{n} \sqrt{|\log |\log \varepsilon||}$  in Theorem 1.1. Integrating methods of [48] and/or [4] should improve some of these bounds. The second issue arises from the poor bounds on the intervortex distance for the ODE

(1.11). Better knowledge of how the ODE behaves should improve the vortex density allowed here.

Although (1.1) provides a fertile ground to test the mathematics of the Gorkov-Eliashberg equations, the more physical problem entails looking at the hydrodynamic limit of (1.4). For the Gorkov-Eliashberg equations (1.4), corresponding proofs of the vortex motion law are due to the second author [51] for  $O(1)$  fields and Sandier-Serfaty [43] for larger fields, following the formal asymptotic work of [40]. Formally, it was shown by Chapman-Rubinstein-Schatzman [7] that the hydrodynamic limit of the associated ODE arising from the vortex motion law of (1.4) converges to a weak solution of

$$\begin{aligned}\partial_t \omega + \operatorname{div}(\omega v) &= 0 \\ v &= \nabla (\Delta - \mathbb{I})^{-1} \omega.\end{aligned}$$

A rigorous approach of the hydrodynamic limit of (1.4) will be studied in [29].

For these problems, there has been a lot of recent progress on the limiting equations for the vortex densities. Ambrosio-Serfaty [1] and Ambrosio-Mainini-Serfaty [2] study them as a metric gradient flow in the space of measures, with the Wasserstein distance as the natural metric. However, they do not obtain the convergence. Even when it becomes possible to carry out the program outlined in the survey of Serfaty [49] and to directly obtain the Wasserstein gradient flow studied in [1, 2] from the Gorkov-Eliashberg equation by a  $\Gamma$ -convergence of gradient flows type result, we believe that our approach will still be useful. For one, it provides quantitative bounds that are useful in type II superconductors without going to the  $\varepsilon \rightarrow 0$  limit of “extreme” type II superconductivity. More importantly, as our approach does not rely on the gradient flow structure, it can be adapted to yield results for more general situations, such as the mixed flows studied in [26] and [39] for the ungauged problem and in [27] and [50] for the gauged problem. Such motion laws have physical importance, as they can be used to explain the sign-change in the Hall effect of type II superconductors, see [11], [24]. Similarly, we expect that our approach can be adapted also to the Hamiltonian Ginzburg-Landau wave system, where results for the PDE to ODE limit for finitely many vortices have been found in [18] and [31], and the ODE to mean field PDE limit has been studied in [35].

**1.4. Method.** We finish the introduction with an outline of the arguments in the paper. The general scheme of the paper is to carefully deduce the vortex motion law for the time dependent Ginzburg-Landau equations by energetic methods. These methods, which in many ways are not as strong as the elliptic PDE approach of Serfaty [47, 48] or the parabolic PDE approach of Bethuel-Orlandi-Smetts [4], do provide a way to avoid using convergence properties too early in the proof. In particular a fundamental feature of the arguments of all previous vortex motion law proofs in (1.1) is to use a limiting kinetic energy lower bound. This bound proves to be difficult to carry to the problem of asymptotically large number of vortices. Furthermore, though our setting provides weaker control on the rate of convergence of  $u$  to  $u_\star$  since we do not take advantage of strong convergence; nonetheless, we are still able to establish rates of convergence and long time of existence at the finite  $\varepsilon$  level. Furthermore, since we are less reliant on strong convergence, our approach is more applicable to



situations where parabolic/elliptic regularization fails, such as in the mixed Ginzburg-Landau equation  $\frac{1}{|\log \varepsilon|} \partial_t u + i \partial_t u = \Delta u + \frac{1}{\varepsilon^2} u (1 - |u|^2)$  and the Ginzburg-Landau wave equation  $\frac{1}{|\log \varepsilon|} \partial_t^2 u = \Delta u + \frac{1}{\varepsilon^2} u (1 - |u|^2)$ . Our approach follows the program of the second author and R. Jerrard [23] for the Gross-Pitaevsky equation  $i \partial_t u = \Delta u + \frac{1}{\varepsilon^2} u (1 - |u|^2)$ . Surprisingly implementing this approach for (1.1) is more challenging and requires several new estimates including an explicit estimate relating the kinetic energy of the PDE to the kinetic energy of the ODE in Section 6 that are not needed in the Gross-Pitaevsky problem.

In Section 2 we provide some background on the renormalized energy and discuss the differential identities on  $|u|$ ,  $j(u)$ , and  $e_\varepsilon(u)$ . Finally, we end the section with a list of some asymptotic bounds on fundamental quantities. These asymptotic bounds will either be prescribed through the choice of initial data or arise as a consequence of the results in the paper. Though the asymptotic bounds seem somewhat arbitrarily chosen, they represent a class of data for which we can close the arguments. Though there is room for improvement in the choice of these asymptotic rates, the order of magnitude of each choice is constrained by our arguments.

We then turn our attention in Section 3 to a discussion of how to pin the vortex positions in (1.1) to the ODE positions. At each time instant one can prove, via localization estimates discussed in Section 5, that  $\frac{e_\varepsilon(u)}{|\log \varepsilon|}$  is close to a sum of delta functions with the error controlled by the amount of the excess energy. However, despite knowing the location up to a bunch of point locations  $\xi_j$ , it is not known *a priori* how close the  $\xi_j$ 's are to the  $a_j$ 's, the expected ODE locations. One could try to establish control on a quantity like  $\sum_{j=1}^n |\xi_j - a_j|$  via a Gronwall inequality. Though this approach works in the  $\varepsilon \rightarrow 0$  limit, see for instance [33, 8, 18, 31], it does not work well for the fixed  $\varepsilon$  problem. In particular one does not know if the  $\xi_j$ 's are Lipschitz at the fixed  $\varepsilon$  level. As in [23] we then introduce a weaker measure  $\eta = \sum |\frac{e_\varepsilon(u)}{|\log \varepsilon|} \chi(x - a_j)(x - a_j)|$  for a suitable cutoff function  $\chi$ . This function has the same effective behavior but is differentiable. At this point we define a sequence of timescales on which  $\eta$  is well behaved in different senses. On these timescales it is shown that  $\eta(t)$  is close to  $\sum_j |\xi_j(t) - a_j(t)|$ , along with other canonical measures. Therefore, our game plan is to prove that  $\eta(t)$  is small on these define timescales via a Gronwall inequality; and hence, pin the vortices to the  $a_j$ 's.

Section 4 begins the Gronwall argument by computing  $\frac{d}{dt} \eta$ , and this computation yields several terms that can be directly estimates found in Sections 5-6. We remark that this line of attack works only for some of the terms at arise from the time derivative of  $\eta$ . Several terms arise that can only be estimated by  $C\sqrt{\eta}$ , which creates havoc within the Gronwall argument. Rather we will later look at a Gronwall argument for a time-averaging of  $\eta$ , and this proves to be sufficient when we combine this with a Lipschitz bound on  $\eta$  proved at the end of this section.

Section 5 contains several technical results, a few of which arise out [23], on the Ginzburg-Landau energy. The first result state that if  $J(u)$  is close to a sum of delta functions located at  $\alpha = \{\alpha_j\}$ , then we can find an even better collection of vortex locations  $\xi = \{\xi_j\}$  that localize  $J(u)$  up to a small error that depends on the excess energy. The second result in this section provides a rate of  $\dot{H}^1$  convergence of  $u$  to

$u_\star = u_\star(\alpha)$  outside of the vortex cores with an explicit errors that depends on the localization of the Jacobian to the  $\alpha = \{\alpha_j\}$  and the excess energy. These first two results are established in [23] based on refined Jacobian estimates and depend heavily on results from [22]. The third result is a new estimate on the localization of the Ginzburg-Landau energy density to the same set of delta functions

$$\left\| \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \sum_{j=1}^n \pi \delta_{\xi_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \lesssim \frac{1}{|\log \varepsilon|} [nD(a) + W(a)].$$

The estimate presented here is a refined (i.e.  $\varepsilon$ -rate dependent) version of an estimate found in Colliander-Jerrard [8].

Since the localization and gamma stability error estimates depend explicitly on the excess energy, it is necessary to understand how the excess energy evolves in time. By the energy dissipation identities (1.9) and (1.12) we see that,

$$D(a(t)) = D(a(0)) + \int_0^t |\dot{a}|^2 - \int_0^t \int_\Omega \frac{|\partial_t u|^2}{\pi |\log \varepsilon|};$$

consequently,  $D(a(t))$  can be controlled by well-preparedness of the initial data and a lower bound on the kinetic energy. The main result of Section 6, and hence on the excess energy, is Theorem 6.1 which provides the lower bound

$$\int_0^t \int_\Omega \frac{|\partial_t u|^2}{\pi |\log \varepsilon|} - \int_0^t |\dot{a}|^2 \geq -\mathbf{error},$$

where **error** depends explicitly on  $\varepsilon$ ,  $n$ ,  $\rho_\star$ ,  $T$ , and  $\sup_{0 \leq t \leq T} |\eta(t)|$ . This result provides a purely quantitative approach to the kinetic energy lower bounds that are found in [31, 18, 43], each of which rely on compactness properties to get a lower bound. To establish this result we make quantitative the kinetic energy estimate of [31], who used the differential identity for the conservation law along with an equipartition of energy result. Here we make use of an optimal equipartitioning result in [28] that identifies how close the tensor  $\nabla u \otimes \nabla u$  is to the diagonal matrix  $|\partial_j u|^2 \delta_{jk}$  ( $\delta_{jk}$  being the kroenecker delta function). Placing this equipartitioning result into the differential identity for the Ginzburg-Landau energy  $e_\varepsilon(u)$ , applying a test function

$$\sum \chi(x - a_j(t)) \dot{a}_j(t) \cdot (x - a_j(t)), \quad \chi \text{ a smooth cutoff function,}$$

and integrating over  $\Omega$  yields the lower bound.

Though we are able to estimate most terms arising in our differential identity for  $\frac{d}{dt}\eta$  in Section 4 by  $\eta$  or by a small error depending on  $\varepsilon$ , there are several terms that have the general form  $\int_{\Omega_\rho} \zeta \left( \frac{j(u)}{|u|} - j(u_\star) \right)$  for some explicit function  $\zeta$ . A naive bound on such terms yields an estimate of the type  $\sqrt{\eta}$  which, unfortunately, destroys the sensitive Gronwall argument. Instead we prove that these terms are small directly by showing  $\frac{j(u)}{|u|} - j(u_\star)$  is small on the time scale that we are considering. This is achieved by employing a Hodge decomposition  $\frac{j(u)}{|u|} - j(u_\star) = \nabla f_1 + \nabla^\perp f_2$ . We show  $f_1$  is small by the localization estimate on the Jacobian and  $f_2$  is small via an involved time-averaging argument. This method was employed in [23] in the Gross-Pitaevsky equations via a time-averaging technique. In Section 7 we prove an analogue of this

type of bound for (1.1) for the parabolic problem. Unfortunately, since we have a different differential identity for  $j(u)$  and weaker bounds on many intermediate results, we feel it necessary to prove this proposition in detail for the sake of completeness.

Finally we complete the proof of the motion in Section 8. This entails showing that we can extend the time-scales introduced in Section 2 to a long enough time  $T$  claimed in Theorem 1.1. In particular we show by a Gronwall argument that the time-averaged  $\eta$  behaves well on the long time scale, so long as vortices do not get too close.

At this point we turn our attention to the hydrodynamic limit. Our first priority is to build a suitably large class of initial data for which there are asymptotically large number of vortices  $n \rightarrow \infty$  as  $\varepsilon \rightarrow 0$  and for which this class of initial data satisfies the ODE for times on the order of at least  $O(\frac{1}{n})$ . We note that for (1.1) with Neumann boundary conditions, vortices will not remain away from the boundary of the domain for long enough times. Therefore, we turn our attention to the Dirichlet problem in Section 9. By a careful consideration of the renormalized energy  $W(a)$  we show that vortices obey the same logarithmic repulsion at the boundary that they feel from each other. This implies a long enough time of existence of the ODE (1.11). Initial data  $u(0)$  with vortices satisfying the conditions prescribed for our class of ODE initial data can then survive on a long enough time scale for our hydrodynamic limit.

In Section 10 we complete the proof Theorem 1.3 on the hydrodynamic limit. The first part of the section we proof an analogue of results of [46, 36] for the Euler point vortex method on  $\mathbb{R}^2$  and the gradient flow version of [34] on  $\mathbb{R}^2$  for bounded domains. By a careful expansion of the time dependent behavior of  $\frac{1}{n} \sum_{j=1}^n \delta_{a_j(t)}$  integrated against a test function with compact support. By using estimates on the Neumann function and implementing the strategy of [36, 34] we prove the convergence and the local energy bound. To complete the proof of the theorem, we show that nonnegative vortex sheet initial with compact support can be approximated by a sequence of a sum of degree-one vortices that satisfy the conditions of our class of initial data. The novel approximation scheme leads to an asymptotically long time of existence with asymptotically large numbers of vortices. Finally, we set  $\omega_n(t) = \frac{1}{n} \frac{e_\varepsilon(u(t))}{\pi |\log \varepsilon|}$ , which has similar behavior as the vortex density function  $\frac{1}{n} \sum_{j=1}^n \delta_{a_j(t)}$  due to Theorem 1.1. Relating this density to the ODE hydrodynamic limit, then achieves the desired hydrodynamic limit.

## 2. NOTATION, ASSUMPTIONS, AND SOME IDENTITIES

In this section we provide some notation and background on the renormalized energy. We then list some differential identities on the supercurrent and energy density. Finally, we discuss some assumptions that we will want our solutions of (1.1) to satisfy.

### 2.1. Renormalized and approximate energies. Recall

$$\Omega_\rho(\alpha) = \Omega \setminus \bigcup_{j=1}^n B_\rho(\alpha)$$

and  $\Omega_\rho$  when the center of the balls is unambiguous from the context. Recall from [3] the canonical harmonic map  $u_\star$  which satisfies the following Hodge system

$$\operatorname{div} j(u_\star) = 0 \quad \operatorname{curl} j(u_\star) = 2\pi \sum_{j=1}^n \delta_{a_j}$$

with either

$$j(u_\star) \cdot \tau = n\partial_\tau \theta + \partial_\tau \varphi$$

on  $\partial\Omega$  or

$$j(u_\star) \cdot \nu = 0$$

on  $\partial\Omega$ . Furthermore, there exists a  $G$  with  $\nabla^\perp G = j(u_\star)$ , where  $G(x) = G(x; \alpha)$  is defined by the following Poisson equation

$$(2.1) \quad \Delta G = 2\pi d_j \sum_{j=1}^n \delta_{\alpha_j} \text{ in } \Omega$$

with either

$$\partial_\nu G = n\partial_\tau \theta + \partial_\tau \varphi \text{ on } \partial\Omega$$

or

$$G = 0 \text{ on } \partial\Omega.$$

The *renormalized energy* then is defined as

$$W(\alpha, d) = \lim_{\rho \rightarrow 0} \left( \int_{\Omega_\rho(\alpha)} \frac{1}{2} |\nabla u_\star|^2 - \pi n \log \frac{1}{\rho} \right).$$

We also define the *approximate energy* as

$$W_\varepsilon(\alpha, d) = W(\alpha, d) + n(\pi |\log \varepsilon| + \gamma)$$

where  $\gamma$  is the constant of [3]. Finally, the *excess energy* is defined as

$$(2.2) \quad D(u; \alpha, d) = E_\varepsilon(u) - W_\varepsilon(\alpha, d).$$

As  $u$  and  $d$  are assumed fixed, we usually simplify this to  $D(\alpha)$  when the context is unambiguous.

We start with a characterization of the gradient of the renormalized energy.

**Lemma 2.1** (Lin [30], Jerrard-Soner [21]). *Let  $\xi \in \Omega^{n*}$  and  $d \in \{\pm 1\}$  then the canonical harmonic map  $u_\star = u_\star(\cdot; \xi, d)$  and the renormalized energy  $W(\xi, d)$  satisfy*

$$(2.3) \quad \int \partial_{x_k x_m} \varphi \left[ (j(u_\star))_m (j(u_\star))_l - \frac{1}{2} \delta_{km} |j(u_\star)|^2 \right] = -\frac{1}{\pi} \sum_{j=1}^n d_j \partial_k \varphi(\xi_j) (\nabla_{\xi_j} W(\xi, d))_k$$

where  $\varphi \in C^2(\Omega)$  and  $\nabla^2 \varphi$  has support in a neighborhood of the  $\xi_j$ 's.

Next we list estimates on the canonical harmonic map and renormalized energy and their derivatives.

**Lemma 2.2.** *There exists constants  $C$  depending only on  $\Omega$  such that for every bounded, open  $\Omega \subset \mathbb{R}^2$ ,  $a \in \Omega^{n*}$  and  $d \in \{\pm 1\}^n$ , the renormalized energy  $W(a, d)$ , canonical harmonic map  $u_\star(\cdot; a, d)$  and its potential  $G(\cdot; a, d)$  as defined in (2.1) satisfy*

$$(2.4) \quad \|j(u_\star)\|_{L^\infty(\Omega_r(a))} = \|\nabla G\|_{L^\infty(\Omega_r(a))} \leq \frac{2n}{r}$$

for all  $r \leq \rho_a$ , and

$$(2.5) \quad |\nabla_i W(a, d)| \leq \frac{Cn}{\rho_a}, \quad |\nabla_i \nabla_j W(a, d)| \leq \frac{Cn}{\rho_a^2}$$

for every  $i, j \in \{1, \dots, n\}$ .

We also have the bound

$$(2.6) \quad W(a, d) \leq C(n^3 + \frac{n^2}{\rho_a^2}).$$

Finally, let  $\xi = (\xi_1, \dots, \xi_n)$  and  $\xi' = (\xi'_1, \dots, \xi'_n)$  with  $\xi, \xi' \in \Omega^{n*}$ . Let  $\Omega_r(\xi, \xi') = \Omega \setminus \left( \bigcup_{j=1}^n B_r(\xi_j) \cup B_r(\xi'_j) \right)$ , then for every  $d \in \{\pm 1\}^n$ ,

$$(2.7) \quad \|j(u_\star)(\xi, d) - j(u_\star)(\xi', d)\|_{L^\infty(\Omega_r(\xi, \xi'))} \leq \frac{1}{r^2} \sum_{j=1}^n |\xi_j - \xi'_j|$$

for all  $r \leq \min\{\rho_\xi, \rho_{\xi'}\}$ , and additionally, for  $1 < p < 2$ ,

$$(2.8) \quad \|j(u(\xi)) - j(u(\xi'))\|_{L^p(\Omega)} \leq (\pi \sum |\xi_i - \xi'_i|)^{\frac{2}{p}-1} (2n\pi)^{2-\frac{2}{p}}.$$

*Proof.* The Neumann boundary condition results are proved in [23]. Corresponding results for the Dirichlet boundary condition can be established by using similar arguments along with estimates of the corresponding renormalized energy found in [47]. Further estimates in the Dirichlet case will be discussed in Section 9.  $\square$

**2.2. Differential identities.** The following identities hold for smooth solutions of (1.1):

$$(2.9) \quad \text{mass identity} \quad \left[ \frac{1}{|\log \varepsilon|} \partial_t - \Delta - \frac{2}{\varepsilon^2} |u|^2 \right] (|u|^2 - 1) = 2 |\nabla u|^2$$

$$(2.10) \quad \text{supercurrent identity} \quad \frac{1}{|\log \varepsilon|} (iu, \partial_t u) = \operatorname{div} j(u)$$

$$(2.11) \quad \text{energy identity} \quad \partial_t e_\varepsilon(u) = \operatorname{div} (u_t, \nabla u) - \frac{|\partial_t u|^2}{|\log \varepsilon|}$$

For fixed  $\varepsilon$  regularity follows from standard parabolic theory. We remark that (2.9) can be used to show that  $0 \leq |u| \leq 1$ , (2.10) will be used to show that  $j(u)$  is nearly divergence-free in a time-averaged sense, and (2.11) is the primary tool to establish the vortex motion law.

**2.3. Assumptions.** We construct initial data that ensures that solutions to (1.1) with the following bounds on fundamental quantities. Though values certainly are not the optimal choices, they do represent values for which we can close in our argument.

$$(2.12) \quad \text{number of vortices} \quad n \leq |\log \varepsilon|^{\frac{1}{200}}$$

$$(2.13) \quad \text{minimal intervortex distance} \quad \rho_\star \geq |\log \varepsilon|^{-\frac{1}{100}}$$

$$(2.14) \quad \text{total time scale} \quad T \leq |\log |\log \varepsilon||$$

$$(2.15) \quad \text{time averaging scale} \quad \delta_\varepsilon = |\log \varepsilon|^{-\frac{1}{4}}$$

$$(2.16) \quad \text{resolution of vortex location} \quad \mathcal{D}_\varepsilon \leq |\log \varepsilon|^{-\frac{1}{4}}$$

$$(2.17) \quad \text{initial excess energy} \quad D(a(0)) \leq |\log \varepsilon|^{-\frac{2}{5}}$$

Given these choices, we will establish a few composite estimates in the next section.

### 3. MEASURING TRUE VORTEX POSITIONS

We define a series of time intervals on which our function  $u$  is well-behaved in different senses. In Subsection 3.1 we will define  $\eta(t)$ , which is a natural measure of the distance of the vortex positions to the  $a_j$ 's.

$$(3.1) \quad \tau_0 = \inf \{t > 0 : \rho_{a(t)} \leq \rho_\star\}$$

$$(3.2) \quad \tau_1 = \sup_t \left\{ 0 \leq t \leq \tau_0 \text{ such that } \|J(u(s)) - \sum_{i=1}^n \pi d_i \delta_{a_i(s)}\|_{\dot{W}^{-1,1}(\Omega)} \leq \frac{C_\star}{|\log \varepsilon|^{\frac{7}{200}}} \right. \\ \left. \text{and } D(a(s)) \leq 100 \text{ for all } 0 \leq s \leq t \right\}$$

$$(3.3) \quad \tau_2 = \sup_t \left\{ 0 \leq t \leq \tau_1 \text{ such that } \eta(s) \leq \frac{1}{2} \mathcal{D}_\varepsilon \text{ for all } 0 \leq s \leq t \right\}$$

$$(3.4) \quad \tau_3 = \sup_t \left\{ \delta_\varepsilon \leq t \leq \tau_2 \text{ such that } \langle \eta \rangle_{\delta_\varepsilon}(s) \leq \frac{1}{4} \mathcal{D}_\varepsilon \text{ for all } \delta_\varepsilon \leq s \leq t \right\}.$$

We note that  $C_\star$  is introduced in Proposition 5.2. By the end of Section 8 we will show that these time intervals overlap, at least until  $T$  defined in (1.15).

The definition of  $\tau_1$  implies that

$$(3.5) \quad \rho_{a(t)} \geq \rho_\star \geq |\log \varepsilon|^{-\frac{1}{100}}, \quad \|J(u) - \sum_{i=1}^n \pi d_i \delta_{a_i(s)}\|_{\dot{W}^{-1,1}(\Omega)} \leq |\log \varepsilon|^{-\frac{7}{200}}$$

for all  $t \in [0, \tau_1]$ , when  $\varepsilon$  is sufficiently small. Therefore when this holds there exist  $\xi(t) = (\xi_1(t), \dots, \xi_n(t)) \in \Omega^{n^\star}$  such that  $|\xi_i - a_i| \leq \frac{\rho_{a(t)}}{4}$  for all  $i$ , and

$$(3.6) \quad \|J(u)(s) - \sum_{i=1}^n \pi d_i \delta_{\xi_i(s)}\|_{\dot{W}^{-1,1}(\Omega)} \leq s_\varepsilon \\ \left\| \frac{e_\varepsilon(u)(s)}{|\log \varepsilon|} - \sum_{i=1}^n \pi \delta_{\xi_i(s)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq t_\varepsilon,$$

where here and throughout this proof,

$$(3.7) \quad \begin{aligned} s_\varepsilon &:= C\varepsilon \left[ \frac{n^5}{\rho_\star} + E_\varepsilon(u_0) \right], \\ t_\varepsilon &:= \frac{C}{|\log \varepsilon|} \left[ n \log \frac{n^4}{\rho_\star} + W(a(0)) \right]. \end{aligned}$$

Those bounds are found in Proposition 5.2 and Theorem 5.3.

We bound the following composite values based on the time interval definitions.

**Lemma 3.1.** *Assuming (2.12)-(2.17) then for  $0 \leq t \leq \tau_1$*

$$(3.8) \quad W(a(t)) \lesssim |\log \varepsilon|^{\frac{3}{100}},$$

$$(3.9) \quad E_\varepsilon(u(t)) \lesssim |\log \varepsilon|^{1+\frac{1}{200}},$$

$$(3.10) \quad s_\varepsilon \lesssim \varepsilon^{\frac{9}{10}},$$

$$(3.11) \quad t_\varepsilon \lesssim |\log \varepsilon|^{-\frac{97}{100}}.$$

*Proof.* By (2.6), since  $\rho_{a(t)} \geq \rho_\star$  then (3.8) follows from (2.12) and (2.13). Next note that

$$\begin{aligned} E(u(t)) &= W_\varepsilon(a(t)) + D(a(t)) = n(\pi |\log \varepsilon| + \gamma) + W(a(t)) + D(a(t)) \\ &\lesssim |\log \varepsilon|^{1+\frac{1}{200}} \end{aligned}$$

from (2.12), (3.8), and the fact that  $D(a(t)) \leq 100$  for  $0 \leq t \leq \tau_1$ . Finally, (3.10) and (3.11) follow from (2.12), (2.13), (3.8), and (3.9).  $\square$

**3.1. Definition and properties of  $\eta(t)$ .** Since the energy is concentrating at the points  $\xi_j$  and the ODE gives us vortex positions  $a_j$ , the main objective of our theorem is to estimate and control  $|\xi(t) - a(t)|$ . This is a challenging quantity to work with directly, so following Jerrard-Spohn [23], we define a similar quantity that is differentiable and has very similar properties. However, in [23] the authors use the Jacobian instead of the energy density due to the differing conservation laws for the Gross-Pitaevsky equation. In the parabolic problem here we set

$$(3.12) \quad \eta(t) := \sum_{j=1}^n |\eta_j(t)| := \sum_{j=1}^n \left| \frac{e_\varepsilon(u)}{|\log \varepsilon|} \Phi_j \right|$$

where

$$\Phi_j(x, t) = \varphi(x - a_j(t)), \quad \varphi(x) = x \chi_{\rho_\star}(x)$$

and  $\chi_{\rho_\star}(x) = \chi(\frac{x}{\rho_\star})$  for a fixed  $\chi \in C_0^\infty(\mathbb{R}^2)$  satisfying  $\chi(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}$ . The  $\Phi_j$ 's are supported on  $B_{2\rho_\star}(a_j(t))$ , so that  $\{\text{supp } \Phi_j(x, t)\}$  are pairwise disjoint when  $\rho_{a(t)} \geq \rho_\star$  and in particular for all  $0 \leq t \leq \tau_1$ .

We claim

**Lemma 3.2.** *If  $0 \leq t \leq \tau_1$  then*

$$\begin{aligned} \left| \eta(t) - \sum_i \pi |\xi_i(t) - a_i(t)| \right| &\lesssim t_\varepsilon \\ \left| \eta(t) - \left\| \frac{e(u(t))}{|\log \varepsilon|} - \sum_{i=1}^n \pi \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \right| &\lesssim t_\varepsilon \\ \left| \eta(t) - \left\| J(u(t)) - \sum_{i=1}^n \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \right| &\lesssim t_\varepsilon \end{aligned}$$

*Proof.* First note that in view of, the definition of  $\tau_1$ , and

$$\begin{aligned} (3.13) \quad \pi \sum_j |\xi_j(t) - a_j(t)| &= \left\| \sum_{i=1}^n \pi d_i (\delta_{\xi_i(t)} - \delta_{a_i(t)}) \right\|_{\dot{W}^{-1,1}(\Omega)} \\ &\lesssim |\log \varepsilon|^{-\frac{7}{200}} + t_\varepsilon \leq \frac{\rho_\star}{4} \end{aligned}$$

when  $\varepsilon$  is sufficiently small, for all  $t \in [0, \tau_1]$ . From the definition of  $\Phi_j$  it follows that  $\xi_j(t) - a_j(t) = \Phi_j(\xi_j(t), t)$  for all such  $t$ . Therefore, there exists a unit vector  $v_j(t)$  such that  $|\xi_j(t) - a_j(t)| = v_j \cdot (2d_j \Phi_j(\xi_j(t)))$ ; hence,

$$\begin{aligned} (3.14) \quad &\pi \sum_j |\xi_j(t) - a_j(t)| \\ &= \int \left( \pi \sum d_i \delta_{\xi_i(t)} \right) \left( \sum v_j \cdot \Phi_j(t) \right) \\ &\leq \eta(t) + \int \left( \pi \sum \delta_{\xi_i(t)} - \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} \right) \left( \sum v_j \cdot \Phi_j(t) \right) \\ &\leq \eta(t) + \left\| \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} - \pi \sum \delta_{\xi_i(t)} \right\|_{\dot{W}^{-1,1}} \left\| \sum_j v_j \cdot \Phi_j(t) \right\|_{W^{1,\infty}} \\ &\leq \eta(t) + Ct_\varepsilon \text{ for all } t \in [0, \tau_1]. \end{aligned}$$

A similar argument shows that for such  $t$ ,

$$(3.15) \quad \eta(t) \leq \pi \sum |\xi_i(t) - a_i(t)| + Ct_\varepsilon.$$

Again following the idea in [23] we use the triangle inequality and the  $\dot{W}^{-1,1}$  norm as the length of a minimal connection to get

$$\begin{aligned} (3.16) \quad &\left\| \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} - \sum_{i=1}^n \pi \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \\ &\leq \left\| \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} - \sum_{i=1}^n \pi \delta_{\xi_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} + \left\| \sum_{i=1}^n \pi d_i (\delta_{\xi_i(t)} - \delta_{a_i(t)}) \right\|_{\dot{W}^{-1,1}(\Omega)} \\ &\leq t_\varepsilon + \pi \sum |\xi_i(t) - a_i(t)| \\ &\leq Ct_\varepsilon + 2\eta(t) \quad \text{for all } t \in [0, \tau_1]. \end{aligned}$$



In the same way one finds that

$$(3.17) \quad \eta(t) \leq Ct_\varepsilon + \left\| \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} - \sum_{i=1}^n \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \quad \text{for all } t \in [0, \tau_1].$$

In particular this implies that when  $\varepsilon$  is sufficiently small,

$$(3.18) \quad \eta(t) \leq 2\mathcal{D}_\varepsilon \quad \text{for all } t \in [0, \tau_1].$$

Finally we can follow a similar argument to show the last estimate.  $\square$

#### 4. INITIALIZATION OF GRONWALL ARGUMENT

Use the notation  $\eta_j(t) := \int \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} \Phi_j(x, t) \in \mathbb{R}^2$  as introduced above, then we have

**Lemma 4.1.** *Let  $u$  be a solution to (1.1). Then for  $0 \leq t \leq \tau_1$  and  $j = 1, \dots, n$*

$$(4.1) \quad \dot{\eta}_j = T_{j,1} + T_{j,2} + T_{j,3} + T_{j,4} + T_{j,5} + T_{j,6} + T_{j,7}$$

where

$$\begin{aligned} T_{j,1} &= \nabla \varphi(\xi_j - a_j) \cdot \mathbb{J}(\nabla_j W(\xi) - \nabla_j W(a)) \\ T_{j,2} &= - \int \left( \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \sum_{i=1}^n \pi \delta_{\xi_i} \right) \left( -\frac{1}{\pi} \nabla_j W(a) \right) \cdot \nabla \varphi(x - a_j) \\ T_{j,3} &= \int \partial_{x_k x_\ell}^2 \Phi_j (\partial_{x_\ell} |u|, \partial_{x_k} |u|) - \partial_{x_\ell x_\ell}^2 \Phi_j e_\varepsilon(|u|) \\ T_{j,4} &= \int \partial_{x_k x_\ell}^2 \Phi_j \left[ \left( \frac{j(u)}{|u|} - j(u_\star) \right)_\ell \left( \frac{j(u)}{|u|} - j(u_\star) \right)_k - \frac{\delta_{k\ell}}{2} \left| \frac{j(u)}{|u|} - j(u_\star) \right|^2 \right] \\ T_{j,5} &= \int \partial_{x_k x_\ell}^2 \Phi_j \left( \frac{j(u)}{|u|} - j(u_\star) \right)_\ell (j(u_\star))_k \\ T_{j,6} &= \int \partial_{x_k x_\ell}^2 \Phi_j \left( \frac{j(u)}{|u|} - j(u_\star) \right)_k (j(u_\star))_\ell \\ T_{j,7} &= - \int \partial_{x_k x_k}^2 \Phi_j \left( \frac{j(u)}{|u|} - j(u_\star) \right)_\ell (j(u_\star))_\ell. \end{aligned}$$

*Proof.* Differentiating  $\eta_j$ , we obtain

$$\frac{d}{dt} \eta_j = \int \frac{e_\varepsilon(u)}{|\log \varepsilon|} \frac{d}{dt} \Phi_j + \int \Phi_j \frac{d}{dt} \frac{e_\varepsilon(u)}{|\log \varepsilon|}$$

Since  $\frac{d}{dt} \Phi_j(x, t) = \frac{d}{dt} \varphi(x - a_j) = (-\dot{a}_j) \cdot \nabla \varphi(x - a_j)$ , we can use the ODE and the fact that  $\Phi_j(\xi_i(t)) = 0$  for  $i \neq j$  to write

$$\begin{aligned} \int \frac{e_\varepsilon(u)}{|\log \varepsilon|} \frac{d}{dt} \Phi_j &= \int \frac{e_\varepsilon(u)}{|\log \varepsilon|} (-\dot{a}_j) \cdot \nabla \varphi(x - a_j) \\ &= -\dot{a}_j \cdot \nabla_j W(a) \cdot \nabla \varphi(\xi_j - a_j) \\ &\quad + \int \left( \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \sum_{i=1}^n \pi \delta_{\xi_i} \right) \left( -\frac{1}{\pi} \nabla_j W(a) \right) \cdot \nabla \varphi(x - a_j) \end{aligned}$$

Next from the evolution identity for the energy, the representation  $\nabla u = \left( \nabla |u| + i \frac{j(u)}{|u|} \right) \frac{u}{|u|}$ , and  $A^2 - B^2 = |A - B|^2 + 2(A - B) \cdot B$  we find

$$\begin{aligned}
\int \Phi_j \frac{d}{dt} \frac{e_\varepsilon(u)}{|\log \varepsilon|} &= \int \Phi_j \partial_{x_k} \frac{(\partial_t u, \partial_{x_k} u)}{|\log \varepsilon|} - \int \Phi_j \frac{|\partial_t u|^2}{|\log \varepsilon|^2} \\
&= \int \partial_{x_k x_\ell}^2 \Phi_j \left[ (\partial_{x_\ell} u, \partial_{x_k} u) - \frac{\delta_{k\ell}}{2} |\nabla u|^2 \right] - \partial_{x_\ell x_\ell}^2 \Phi_j \frac{(1 - |u|^2)^2}{4\varepsilon^2} \\
&\quad - \int \Phi_j \frac{|\partial_t u|^2}{|\log \varepsilon|^2} \\
&= \int \partial_{x_k x_\ell}^2 \Phi_j (\partial_{x_\ell} |u|, \partial_{x_k} |u|) - \partial_{x_\ell x_\ell}^2 \Phi_j e_\varepsilon(|u|) \\
&\quad + \int \partial_{x_k x_\ell}^2 \Phi_j \left[ \left( \frac{j(u)}{|u|} - j(u_\star) \right)_\ell \left( \frac{j(u)}{|u|} - j(u_\star) \right)_k - \frac{\delta_{k\ell}}{2} \left| \frac{j(u)}{|u|} - j(u_\star) \right|^2 \right] \\
&\quad + \int \partial_{x_k x_\ell}^2 \Phi_j \left( \frac{j(u)}{|u|} - j(u_\star) \right)_\ell (j(u_\star))_k \\
&\quad + \int \partial_{x_k x_\ell}^2 \Phi_j \left( \frac{j(u)}{|u|} - j(u_\star) \right)_k (j(u_\star))_\ell \\
&\quad - \int \partial_{x_\ell x_\ell}^2 \Phi_j \left( \frac{j(u)}{|u|} - j(u_\star) \right)_\ell (j(u_\star))_\ell \\
&\quad - \frac{1}{\pi} \sum_{k=1}^n \partial_{x_k}^2 \Phi_j (\nabla_{\xi_k} W(\xi, d))_k.
\end{aligned}$$

□

Next we obtain an estimate of  $\dot{\eta}$  by separately considering contributions from the different terms isolated in Lemma 4.1. We will prove

**Lemma 4.2.** *For  $t \in [0, \tau_1]$*

$$|\dot{\eta}(t)| \lesssim \frac{1}{\rho_\star} \left( \mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + \mathcal{B}_\varepsilon \right) + \frac{n^{\frac{3}{2}}}{\rho_\star} \sqrt{\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + \mathcal{B}_\varepsilon}.$$

*Proof.* The condition  $\varepsilon < \varepsilon_0$  is needed to guarantee the validity of estimates (3.14), (3.17), (6.19) from Section 6.

Note from Lemma 4.1 and the definition (3.12) of  $\eta$  that

$$(4.2) \quad \dot{\eta} = T_1 + \dots T_7, \quad \text{where } T_k = \sum_{j=1}^n \frac{\eta_j}{|\eta_j|} \cdot T_{j,k}.$$

We estimate these terms in turn.

First, note that  $\nabla\phi(\xi_j - a_j) = \xi_j - a_j$  for  $0 \leq t \leq \tau_1$ , by the definition of  $\phi$  and (3.13). Thus, in view of (3.14),

$$|T_1| \leq \sum_j |T_{j,1}| \leq C(\eta + t_\varepsilon) \sum_j |\nabla_j W(\xi) - \nabla_j W(a)|.$$

And arguing as in the proof of (2.7) we see that

$$\begin{aligned} |\nabla_j W(\xi) - \nabla_j W(a)| &\leq \sum_{k=1}^n |\xi_k(t) - a_k(t)| \left( \sup_k \sup_{|y-a(t)| \leq |\xi(t)-a(t)|} |\nabla_k \nabla_j W(y)| \right) \\ &\leq (\eta(t) + Ct_\varepsilon) C \frac{n}{\rho_\star^2}, \end{aligned}$$

using (3.14) again, as well as bounds on  $\nabla^2 W$  from (2.5). Thus

$$(4.3) \quad |T_1| \leq C \frac{n^2}{\rho_\star^2} (\eta(t) + Ct_\varepsilon)^2.$$

Next,

$$\begin{aligned} |T_2| &= \left| \int \left( J(u) - \sum_{i=1}^n \pi d_i \delta_{\xi_i} \right) \left( \sum_j \mathbb{J} \nabla_j W(a) \cdot \nabla(\Phi_j \cdot \frac{\eta_j}{|\eta_j|}) \right) \right| \\ &\leq \left\| J(u) - \sum_i \pi d_i \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \left\| \nabla \sum_j \nabla_j W(a) \cdot \nabla(\Phi_j \cdot \frac{\eta_j}{|\eta_j|}) \right\|_{L^\infty}. \end{aligned}$$

Since the  $\Phi_j$ 's have disjoint support

$$\left\| \nabla \sum_j \nabla_j W(a) \cdot \nabla(\Phi_j \cdot \frac{\eta_j}{|\eta_j|}) \right\|_{L^\infty} \leq \sup_j |\nabla_j W(a)| \|\nabla^2 \Phi_j\|_\infty \leq C \frac{n}{\rho_\star^2}.$$

We conclude from (3.6) and the above that

$$(4.4) \quad |T_2| \leq Cs_\varepsilon \frac{n}{\rho_\star^2}.$$

Continuing, we use the fact that  $\nabla^2 \Phi_j$  vanishes in  $B_{\rho_\star}(a_j)$ , together with (6.19), to find that

$$\begin{aligned} |T_3| &\leq \left\| \sum_j \frac{\eta_j}{|\eta_j|} \cdot \nabla^2 \Phi_j \right\|_{L^\infty} \int_{\Omega_{\rho_\star}(a)} |\nabla|u||^2 \\ (4.5) \quad &\lesssim \frac{1}{\rho_\star} \left[ \mathcal{A}_\varepsilon \sup_{s \in [0,t]} \eta(s) + \mathcal{B}_\varepsilon \right]. \end{aligned}$$

Exactly the same considerations show that

$$(4.6) \quad |T_4| \lesssim \frac{1}{\rho_\star} \left[ \mathcal{A}_\varepsilon \sup_{s \in [0,t]} \eta(s) + \mathcal{B}_\varepsilon \right].$$

Next,

$$|T_5| \leq \left\| \sum_j \frac{\eta_j}{|\eta_j|} \cdot \nabla^2 \Phi_j \right\|_{L^\infty} \left\| \frac{j(u)}{|u|} - j(u_\star) \right\|_{L^2(\Omega_{\rho_\star})} \|j(u_\star)\|_{L^2(\cup_j \text{supp} \nabla^2 \Phi_j)}.$$

Using (2.4), one can easily check that  $\|j(u_\star)\|_{L^2(\cup_j \text{supp} \nabla^2 \Phi_j)} \leq \frac{Cn}{\rho_\star} (Cn\rho_\star^2)^{\frac{1}{2}}$ , and hence we conclude that

$$(4.7) \quad |T_5| \lesssim \frac{n^{\frac{3}{2}}}{\rho_\star} \sqrt{\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + \mathcal{B}_\varepsilon}.$$

Exactly the same argument shows that  $|T_6| + |T_7| \lesssim \frac{n^{\frac{3}{2}}}{\rho_\star} \sqrt{\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + \mathcal{B}_\varepsilon}$ .  $\square$

The result of the lemma is not good enough to get any very strong result from Gronwall's inequality, but it still implies a useful Lipschitz bound. Since we will be using a time averaging operator, we define the *time average* of a function  $h$  as

$$\langle h \rangle_{\delta_\varepsilon}(t) = \frac{1}{\delta_\varepsilon} \int_{t-\delta_\varepsilon}^t h(s) ds$$

for any  $t \geq \delta_\varepsilon$ . This will be used to handle estimates that make use of the differential identity (2.10) to show that the current  $j(u)$  is effectively divergence-free on average, and such estimates.

**Corollary 4.3.** *We have for  $t \in [0, \tau_2]$*

$$(4.8) \quad |\dot{\eta}| \lesssim \frac{n^{\frac{3}{2}}}{\rho_\star} \sqrt{\mathcal{A}_\varepsilon \mathcal{D}_\varepsilon + \mathcal{B}_\varepsilon} \lesssim |\log \varepsilon|^{-\frac{7}{80}} |\log |\log \varepsilon||^{\frac{1}{2}}.$$

*If  $0 \leq t - \delta_\varepsilon \leq s \leq t \leq \tau_2$  then*

$$(4.9) \quad |\eta(s) - \langle \eta(t) \rangle_{\delta_\varepsilon}| \lesssim \delta_\varepsilon \frac{n^{\frac{3}{2}}}{\rho_\star} \sqrt{\mathcal{A}_\varepsilon \mathcal{D}_\varepsilon + \mathcal{B}_\varepsilon} \lesssim |\log \varepsilon|^{-\frac{27}{80}} |\log |\log \varepsilon||^{\frac{1}{2}}$$

*Proof.* Both estimates come directly from (2.12), (2.13), (2.15), Lemma 4.2, and Theorem 6.1.  $\square$

## 5. QUANTITATIVE RATE OF SECOND-ORDER GAMMA CONVERGENCE

In order to quantify the closeness of our heat equation solution to the point vortex model, we need some results that quantify the closeness of  $u$  to the canonical harmonic map  $u_\star$ , based on the excess energy

$$D(\alpha) = \int_{\Omega} e_\varepsilon(u) - W_\varepsilon(\alpha, d).$$

We also define the excess energy in the ball  $B_\sigma(\alpha_j)$  to be

$$D_{B_\sigma(\alpha_j)} = \int_{B_\sigma(\alpha_j)} e_\varepsilon(u) dx - \left( \pi \log \frac{\sigma}{r} + \gamma \right)$$

where  $\gamma$  is the constant from [3]. The following result lets us compute the rate of  $\dot{H}^1$  convergence of  $u$  to the canonical harmonic function, based on the excess energy.

**Proposition 5.1** (Jerrard-Spirn [23]). *Let  $\Omega$  be a bounded, open simply connected subset of  $\mathbb{R}^2$  with  $C^1$  boundary. Then there exists constants  $C, K_1$  depending only on  $\Omega$  such that for any  $u \in H^1(\Omega; \mathbb{C})$ , if there exist  $n \geq 0$ , finite, with  $\alpha = (\alpha_1, \dots, \alpha_n) \in \Omega^{n*}$  and  $d \in \{\pm 1\}^n$  such that*

$$(5.1) \quad \left\| J(u) - \sum_{j=1}^n \pi d_j \delta_{\alpha_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq s_\varepsilon \quad \text{for some } s_\varepsilon \in [\varepsilon \sqrt{\log(\rho_\alpha/\varepsilon)}, \rho_\alpha/K_1],$$

and if  $4s_\varepsilon \leq \sigma^* = \sqrt{\frac{\rho_\alpha}{n^3} (s_\varepsilon + \varepsilon E_\varepsilon(u))} \leq \frac{\rho_\alpha}{K_1}$  then

$$(5.2) \quad \int_{\Omega_{\sigma^*}(\alpha)} e_\varepsilon(|u|) + \frac{1}{4} \left| \frac{j(u)}{|u|} - j(u_\star) \right|^2 \leq D(\alpha) + C \sqrt{\frac{n^5}{\rho_\alpha} (s_\varepsilon + \varepsilon E_\varepsilon(u))}$$

for canonical harmonic map  $u_\star$ . Finally,

$$(5.3) \quad \|j(u) - j(u_\star)\|_{L^{4/3}(\Omega)} \leq C \sqrt{D(\alpha)} + \mathbf{error}$$

and

$$\begin{aligned} \mathbf{error} &\leq C \varepsilon^{\frac{1}{2}} E_\varepsilon(u)^{\frac{3}{4}} \\ &\quad + C n (s_\varepsilon + \varepsilon E_\varepsilon(u))^{\frac{1}{4}} \left[ \left( \frac{n}{\rho_\alpha} \right)^{\frac{1}{4}} + \rho_\alpha^{\frac{1}{4}} \left( 1 + \sqrt{\frac{E_\varepsilon(u)}{n^3}} \right) \right]. \end{aligned}$$

We also have the following the result that lets us localize the Jacobian about delta functions.

**Proposition 5.2** (Jerrard-Spirn [23]). *Let  $\Omega$  be a bounded, open, simply connected subset of  $\mathbb{R}^2$  with  $C^1$  boundary. Then there exists constants  $C$  and  $C_\star$ , depending on  $\text{diam}(\Omega)$ , with the following property:*

*For any  $u \in H^1(\Omega; \mathbb{C})$ , if there exist  $n \geq 0$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in \Omega^{n*}$  and  $d \in \{\pm 1\}^n$  such that*

$$(5.4) \quad \left\| J(u) - \sum_{j=1}^n \pi d_j \delta_{\alpha_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq \frac{\rho_\alpha}{8C_\star n^5},$$

and if in addition  $E_\varepsilon(u) \geq 1$  and

$$(5.5) \quad \left[ \frac{n^5}{\rho_\alpha} E_\varepsilon(u) + \frac{n^{10}}{\rho_\alpha^2} \sqrt{E_\varepsilon(u)} \right] \leq \frac{1}{\varepsilon},$$

then there exist  $(\xi_1, \dots, \xi_d) \in \Omega^{n*}$  such that  $|\xi_i - \alpha_i| \leq \frac{\rho_\alpha}{2C_\star n^4}$  for all  $i$ , and

$$(5.6) \quad \begin{aligned} &\left\| J(u) - \pi \sum_{i=1}^n d_i \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}} \\ &\leq C \varepsilon \left[ n(C + D(\alpha))^2 e^{\frac{1}{\pi} D(\alpha)} + (C + D(\alpha)) \frac{n^5}{\rho_\alpha} + E_\varepsilon(u) \right] \end{aligned}$$

where  $D(\alpha) = D(u, \alpha, d) = E_\varepsilon(u) - W_\varepsilon(\alpha, d)$ .

We now state a result that clarifies the convergence of  $\frac{e_\varepsilon(u)}{|\log \varepsilon|}$  to a set of delta functions.

**Theorem 5.3.** *Let  $u$  satisfy the same hypotheses as Proposition 5.2, then the  $\{\xi_j\}_{j=1}^n$  found in Proposition 5.2 satisfy*

$$(5.7) \quad \left\| \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \sum_{i=1}^n \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq \frac{C}{|\log \varepsilon|} \left[ n(D(\alpha) + C + \log \frac{n^4}{\rho_\alpha}) + W(\alpha, d) \right].$$

Furthermore, for all  $t \in [0, \tau_1]$  then we can estimate

$$(5.8) \quad \left\| \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \sum_{j=1}^n \delta_{\xi_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \lesssim \frac{n^2}{|\log \varepsilon| \rho_\star^2} \lesssim |\log \varepsilon|^{-\frac{97}{100}}.$$

We make precise (and quantitative) an argument found in [8]. The first step is a moment estimate on the Ginzburg-Landau energy about the vortex core.

**Lemma 5.4.** *If  $\|J(u) - \pi \delta_0\|_{\dot{W}^{-1,1}(B_r)} \leq \frac{r}{4}$  and  $\left| \int_{B_r} e_\varepsilon(u) - \pi \ln \frac{r}{\varepsilon} \right| = K_0$  then there exists  $\xi \in B_{r/2}(0)$  and a constant  $C$ , independent of  $K_0$  and  $u$ , such that*

$$(5.9) \quad \int_{B_r} |x - \xi| e_\varepsilon(u) \leq rC(K_0 + 1)$$

*Proof.* By Theorem 1.2' of [22] then there exists  $\xi \in B_{r/2}(0)$  such that for any  $\tau < r - |\xi|$  and  $\varepsilon \leq \sigma < \tau$ ,

$$(5.10) \quad \int_{B_\tau(\xi) \setminus B_\sigma(\xi)} e_\varepsilon(u) \geq \pi \ln \frac{\tau}{\sigma} - K_0 - C$$

From (5.10) we see that

$$(5.11) \quad \int_{B_\tau(\xi) \setminus B_\varepsilon(\xi)} e_\varepsilon(u) \geq \pi \ln \frac{\tau}{\varepsilon} - K_0 - C$$

so that

$$(5.12) \quad \begin{aligned} \int_{B_\varepsilon(\xi)} e_\varepsilon(u) + \int_{B_r(0) \setminus B_\tau(\xi)} e_\varepsilon(u) &\leq \pi \ln \frac{r}{\varepsilon} + K_0 - \pi \ln \frac{\tau}{\varepsilon} + K_0 + C \\ &\leq \pi \ln \frac{r}{\tau} + CK_0 + C \end{aligned}$$

Now we look at the energy in the annular set  $B_{2^{-j}} \setminus B_{2^{-(j+1)}}(\xi) \subset B_r \setminus B_\varepsilon(\xi)$ . In particular

$$\begin{aligned} \int_{B_{2^{-j}} \setminus B_{2^{-(j+1)}}(\xi)} e_\varepsilon(u) &= \int_{B_r} e_\varepsilon(u) - \int_{B_r \setminus B_{2^{-j}}(\xi)} e_\varepsilon(u) - \int_{B_{2^{-(j+1)}}(\xi)} e_\varepsilon(u) \\ &\leq \pi \ln \frac{r}{\varepsilon} + K_0 - \pi \ln \frac{r}{2^{-j}} + C(K_0 + 1) - \pi \ln \frac{2^{-(j+1)}}{\varepsilon} + C(K_0 + 1) \\ &= \pi \ln 2 + 2C(K_0 + 1) = C(K_0 + 1) \end{aligned}$$

Next we prove the claim. If we let  $2^{-M_\varepsilon} = \frac{r}{2}$  and  $2^{-N_\varepsilon} = \varepsilon$  then

$$\begin{aligned} \int_{B_r} |x - \xi| e_\varepsilon(u) &= \int_{B_\varepsilon} |x - \xi| e_\varepsilon(u) + \int_{B_r \setminus B_{r/2}(\xi)} |x - \xi| e_\varepsilon(u) \\ &\quad + \sum_{j=M_\varepsilon}^{N_\varepsilon} \int_{B_{2^{-j}} \setminus B_{2^{-(j+1)}}(\xi)} |x - \xi| e_\varepsilon(u) \\ &\leq r\pi \ln \frac{r}{r/2} C(K_0 + 1) + \sum_{j=M_\varepsilon}^{N_\varepsilon} 2^{-j} C(K_0) \\ &\leq rC(K_0 + 1) \end{aligned}$$

since  $\sum_{j=M_\varepsilon}^{N_\varepsilon} 2^{-j} \leq r$ . □

In order to establish the proof of the theorem, we use the following energy lower bound:

**Lemma 5.5** (Jerrard-Spirn, [22]). *There exists an absolute constant  $C > 0$  such that if  $u \in H^1(B_\sigma)$  satisfies*

$$\|J(u) \pm \pi\delta_0\|_{\dot{W}^{-1,1}(B_\sigma)} \leq \frac{\sigma}{4}$$

then

$$(5.13) \quad 0 \leq D_{B_\sigma} + C \frac{\varepsilon}{\sigma} \sqrt{\log \frac{\sigma}{\varepsilon}} + \frac{C}{\sigma} \|J(u) - \pi\delta_0\|_{\dot{W}^{-1,1}(B_\sigma)}.$$

We now present the

*Proof of Theorem 5.3.* From the proof of Proposition 5.2 in [23], the  $C_\star$  is chosen so that  $\frac{\rho_\alpha}{C_\star} \leq \frac{1}{2}$ . Then choosing  $\sigma = \frac{\rho_\alpha}{n^4 C_\star}$  we find that

$$(5.14) \quad 4s_\varepsilon = \frac{\rho_\alpha}{2C_\star n^5} = \frac{\sigma}{n} \leq \sigma \leq \frac{\rho_\alpha}{nK_1}$$

In particular this implies

$$(5.15) \quad \|J(u) - \pi d_j \delta_{\alpha_j}\|_{\dot{W}^{-1,1}(B_\sigma(\alpha_j))} \leq \frac{\sigma}{4}$$

for  $\sigma$  defined above.

1. Given the choice of  $\sigma$  we claim that the following bounds hold

$$(5.16) \quad -\frac{C}{n} \leq D_{B_\sigma(\alpha_j)} \leq D(\alpha) + C$$

$$(5.17) \quad \int_{\Omega_\sigma} e_\varepsilon(u) \leq D(\alpha) + W(\alpha, d) + C.$$

In order to prove (5.16) we note that (5.15) and Lemma 5.5 imply  $0 \leq D_{B_\sigma^j} + C \frac{\varepsilon}{\sigma} \sqrt{\log \frac{\sigma}{\varepsilon}} + \frac{C}{\sigma} \|J(u) + \pi d_j \delta_{\alpha_j}\|_{\dot{W}^{-1,1}(B_\sigma)} \leq D_{B_\sigma^j} + \frac{C}{n}$ , which follows if

$$\sigma \geq n^2 \varepsilon.$$

To prove the upper bound we use the following inequality that can be found in the proof of Theorem 3 in [23].

$$(5.18) \quad \begin{aligned} D(\alpha) &= \int_{\Omega_\sigma(\alpha)} [e_\varepsilon(u) - e_\varepsilon(u_\star)] + \sum_{j=1}^n D_{B_\sigma(\alpha_j)} + O\left(\left(\frac{n\sigma}{\rho_\alpha}\right)^2\right) \\ &\geq \int_{\Omega_\sigma(\alpha)} e_\varepsilon(|u|) + \frac{1}{4} \left| \frac{j(u)}{|u|} - j(u_\star) \right|^2 + \sum_{j=1}^n D_{B_\sigma(\alpha_j)} - C \end{aligned}$$

if  $\sigma \leq \frac{\rho_\alpha}{n}$ . Thus  $\sum_{i=1}^n D_{B_\sigma(\alpha_i)} \leq D(\alpha) + C$ , and hence  $D_{B_\sigma(\alpha_i)} \leq D(\alpha) + C$  since  $D_{B_\sigma(\alpha_j)} \geq -\frac{C}{n}$  for each  $j \in \{1, \dots, n\}$ . This finishes the proof of (5.16).

For (5.17) we again write  $D(\alpha) = \int_{\Omega_\sigma} e_\varepsilon(u) - e_\varepsilon(u_\star) + \sum D_{B_\sigma(\alpha_i)} + O\left(\left(\frac{n\sigma}{\rho_\alpha}\right)^2\right)$ , so

$$\begin{aligned} \int_{\Omega_\sigma} e_\varepsilon(u) &= D(\alpha) - \sum D_{B_\sigma(\alpha_i)} + \int_{\Omega_\sigma} e_\varepsilon(u_\star) + O\left(\left(\frac{n\sigma}{\rho_\alpha}\right)^2\right) \\ &= D(\alpha) - \sum D_{B_\sigma(\alpha_i)} + W(\alpha, d) + O\left(\left(\frac{n\sigma}{\rho_\alpha}\right)^2\right) \\ &\leq D(\alpha) + W(\alpha, d) + C. \end{aligned}$$

2. From Lemma 5.4, (5.16), and (5.15) there exists a  $\xi_j$  in  $B_{\sigma/2}(\alpha_j)$  for each  $j = \{1, \dots, n\}$  such that

$$(5.19) \quad \int_{B_\sigma(\alpha_j)} |x - \xi_j| e_\varepsilon(u) \leq C\sigma (D(\alpha) + C)$$

Next we choose an arbitrary  $\phi \in W_0^{1,\infty}(\Omega)$  with  $\|\nabla \phi\|_{L^\infty(\Omega)} \leq 1$ . Then

$$\begin{aligned} &\left| \int_{\Omega} \phi \left( \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \sum \delta_{\xi_j} \right) \right| \\ &\leq \left| \int_{\Omega_\sigma} \phi \frac{e_\varepsilon(u)}{|\log \varepsilon|} \right| + \sum_{j=1}^n \left| \int_{B_\sigma(\alpha_j)} \phi \left( \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \delta_{\xi_j} \right) \right| \\ &= A + B. \end{aligned}$$



We first handle  $A$ . Choose any  $x_0 \in \partial\Omega$  then from (5.17)

$$\begin{aligned}
 \left| \int_{\Omega_\sigma} \phi(x) \frac{e_\varepsilon(u)}{|\log \varepsilon|} \right| &= \left| \int_{\Omega_\sigma} (\phi(x) - \phi(x_0)) \frac{e_\varepsilon(u)}{|\log \varepsilon|} \right| \\
 &\leq \int_{\Omega_\sigma} |x - x_0| \frac{e_\varepsilon(u)}{|\log \varepsilon|} \\
 (5.20) \quad &\leq \frac{\text{diam}(\Omega)}{|\log \varepsilon|} [D(\alpha) + W(\alpha, d) + C]
 \end{aligned}$$

Next we estimate  $B$ . Without loss of generality assume  $\alpha_j = 0$  and  $\xi_j = \xi$  and again choosing  $x_0 \in \partial\Omega$ , then

$$\begin{aligned}
 \left| \int_{B_\sigma} \phi \left( \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \delta_\xi \right) \right| &\leq \left| \int_{B_\sigma} (\phi(x) - \phi(\xi)) \frac{e_\varepsilon(u)}{|\log \varepsilon|} \right| + |\phi(\xi)| \left| \int_{B_\sigma} \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \delta_\xi \right| \\
 &\leq \left| \int_{B_\sigma} |x - \xi| \frac{e_\varepsilon(u)}{|\log \varepsilon|} \right| + |\xi - x_0| \left| \int_{B_\sigma} \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \pi \delta_\xi \right| \\
 &= B_1 + B_2
 \end{aligned}$$

From (5.19) we have

$$B_1 \leq \frac{C}{|\log \varepsilon|} \sigma (D(\alpha) + C)$$

Whereas,  $\left| \int_{B_\sigma} e_\varepsilon(u) - \pi |\log \varepsilon| \right| \leq \pi \log \frac{1}{\sigma} + D_{B_\sigma} + \gamma$  implies

$$B_2 \leq \frac{\text{diam}(\Omega)}{|\log \varepsilon|} \left[ \log \frac{1}{\sigma} + D(\alpha) + C \right]$$

Since  $\sigma = \frac{\rho_\alpha}{K_2 n^4} \leq 1$  then

$$B_1 + B_2 \leq \frac{C}{|\log \varepsilon|} \left[ \log \frac{K_2 n^4}{\rho_\alpha} + D(\alpha) + C \right].$$

Combining with the bound on  $A$  yields

$$\left| \int_{\Omega} \phi \left( \frac{e_\varepsilon(u)}{|\log \varepsilon|} - \sum \pi \delta_{\xi_j} \right) \right| \leq \frac{C}{|\log \varepsilon|} \left[ n(D(\alpha) + C + \log \frac{n^4}{\rho_\alpha}) + W(\alpha, d) \right],$$

and since  $n \leq \frac{1}{\rho_\alpha}$  then (5.7) follows. □

## 6. QUANTITATIVE BOUNDS ON THE KINETIC AND EXCESS ENERGY

We now present a kinetic energy bound for fixed  $\varepsilon$ . Similar bounds with errors of the form  $o_\varepsilon(1)$  can be found in [18, 31, 44].

Our method of proof is inspired by the choice of test function found in the proof of Theorem 3 in [25].

**Theorem 6.1.** Fix  $\delta > 0$  and  $t \leq \tau_1$  of (3.2), then for any time interval  $t_1 \leq t \leq t_2 \leq T$  with  $|t_2 - t_1| \gtrsim 1$ ,

$$(6.1) \quad \begin{aligned} \pi \int_{t_1}^{t_2} |\dot{a}|^2 &\leq \int_{t_1}^{t_2} \int_{\Omega} \chi(x - a(t)) \frac{|\partial_t u|^2}{|\log \varepsilon|} \\ &\quad + C \mathcal{A}_\varepsilon \sup_{t \in [t_1, t_2]} \sum_j |\xi_j(t) - a_j(t)| + C \mathcal{B}_\varepsilon \end{aligned}$$

where

$$(6.2) \quad \begin{aligned} \mathcal{A}_\varepsilon &:= \frac{n^2 T}{\rho_\star^3} \lesssim |\log \varepsilon|^{\frac{1}{25}} |\log |\log \varepsilon|| \\ \mathcal{B}_\varepsilon &:= \frac{n^4 T}{\rho_\star^6 |\log \varepsilon|^{\frac{1}{2}}} \lesssim |\log \varepsilon|^{-\frac{21}{50}} |\log |\log \varepsilon|| \end{aligned}$$

and where  $C$  depends only on  $\Omega$ .

Furthermore, if  $D(a(0)) \leq \mathcal{B}_\varepsilon$ , then

$$(6.3) \quad D(a(t)) \lesssim \mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + \mathcal{B}_\varepsilon,$$

and

$$\begin{aligned} \int_{\Omega_{\rho_\star}(\xi)} e_\varepsilon(|u|) + \left| \frac{j(u)}{|u|} - j(u_\star)(\xi(t)) \right|^2 &\lesssim \mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + \mathcal{B}_\varepsilon \\ \|j(u) - j(u_\star)\|_{L^{4/3}(\Omega)} &\lesssim \sqrt{\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s)} + \sqrt{\mathcal{B}_\varepsilon}. \end{aligned}$$

A similar theorem was proved in [28] for a single vortex that stays an  $O(1)$  distance from the boundary for an  $O(1)$  time. Here we prove a much more explicit estimate. The major tool to establishing a finite- $\varepsilon$  bound on the kinetic energy is the following optimal result on the equipartitioning of Ginzburg-Landau energy, which improves related results in [31] and [44].

**Proposition 6.2** (Kurzke-Spirn [28]). Suppose  $\|J(u) - \pi \delta_0\|_{\dot{W}^{-1,1}(B_\sigma)} \leq \frac{\sigma}{4}$  and  $\int_{B_\sigma} e_\varepsilon(u) \leq \pi \log \frac{\sigma}{\varepsilon} + K_0$  then

$$(6.4) \quad \left| \frac{1}{2} \int_{B_\sigma} \begin{pmatrix} |\partial_{x_1} u|^2 & |(\partial_{x_1} u, \partial_{x_2} u)| \\ |(\partial_{x_1} u, \partial_{x_2} u)| & |\partial_{x_2} u|^2 \end{pmatrix} - \begin{pmatrix} \frac{\pi}{2} \log \frac{\sigma}{\varepsilon} & 0 \\ 0 & \frac{\pi}{2} \log \frac{\sigma}{\varepsilon} \end{pmatrix} \right| \leq \sqrt{K_1 \log \frac{\sigma}{\varepsilon}}$$

where  $K_1 = C(C + K_0)e^{K_0/\pi}$  and  $C$  is a universal constant.

We apply this equipartitioning result to the evolution identity for the energy and deduce a rate of convergence for the kinetic energy.

*Proof of Theorem 6.1.* 1. We generally assume  $t \leq \tau_1$ . Per (3.2), we have for every  $s \in [0, t]$  the estimate

$$(6.5) \quad \left\| J(u(s)) - \sum_{j=1}^n \pi \delta_{a_j(s)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq C_\star \frac{\rho_\star}{n^5} = \tilde{s}_\varepsilon$$

Conservation of energy implies

$$(6.6) \quad \int_0^t \int_{\Omega} \frac{|\partial_t u|^2}{|\log \varepsilon|} = E_{\varepsilon}(u_0) - E_{\varepsilon}(u(t)) \\ = D(a(0)) + W_{\varepsilon}(a_0) - W_{\varepsilon}(a(t)) - D(a(t))$$

From (5.2) we see that  $D(a(t)) \geq -C\sqrt{\frac{n^5}{\rho_{\star}}(\tilde{s}_{\varepsilon} + \varepsilon E_{\varepsilon}(u(t)))} > -C$ . We also have from (2.6) that  $W(a_0) - W(a(t)) \leq |W(a_0)| + |W(a(t))| \leq C\left(n^3 + \frac{n^2}{\rho_{\star}^2}\right)$ . Finally,  $D(a(0)) \leq Cn\frac{\varepsilon^2}{\rho_{\star}^2} \leq C$ . Since  $n < \rho_{\star}^{-1}$ , we find that

$$(6.7) \quad \int_0^t \int_{\Omega} \frac{|\partial_t u|^2}{\pi |\log \varepsilon|} \lesssim \frac{n^2}{\rho_{\star}^2}.$$

and from (5.17) and the assumptions on  $D(a(0))$  and  $W(a(s), d)$ , we have that

$$(6.8) \quad \int_{\Omega_{\sigma}} e_{\varepsilon}(u) \lesssim \frac{n^2}{\rho_{\star}^2}$$

for every  $\sigma$  with  $\sigma \geq C\frac{\rho_{\star}}{n^4}$ . For  $\sigma < \rho_{\star}$ ,  $B_{2\sigma}(a_j(s)) \cap B_{2\sigma}(a_k(s)) = \emptyset$  for all  $s \in [0, t]$  unless  $j = k$ , and  $B_{2\sigma}(a_j(s)) \cap \partial\Omega = \emptyset$  for all  $j$ .

2. We now make the following claim. Let  $\chi \in C_c^{\infty}(\mathbb{R}^2)$  be a function such that  $\chi \geq 0$ ,  $\chi \equiv 1$  on  $B_r(0)$ ,  $\chi \equiv 0$  in  $\mathbb{R}^2 \setminus B_{2r}(0)$  and  $|D^k \chi| \leq Cr^{-k}$  for  $k = 1, 2$ . Then there exists a constant such that for any  $b_j \in C^1([0, T]; \mathbb{R}^2)$ ,  $j = 1, \dots, n$ :

$$(6.9) \quad \left| \int_{t_1}^{t_2} \int_{\Omega} \sum_{j=1}^n \chi(x - a_j(t)) b_j(t) \cdot (\partial_t u, \nabla u) + \sum_{j=1}^n \pi \int_{t_1}^{t_2} b_j(t) \cdot \dot{a}_j(t) \right| \\ \lesssim \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \left[ \frac{t_{\varepsilon}}{\sigma} + \sum_{j=1}^n |\xi_j(t) - a_j(t)| \right] \\ + t_{\varepsilon} \frac{T}{\sigma} \left[ \frac{1}{\sigma} \sup_j \|\dot{a}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} + \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \right] \\ + T \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_{t \in [t_1, t_2]} \sum_{j=1}^n |\xi_j(t) - a_j(t)| + \frac{1}{|\log \varepsilon|} \frac{n^2}{\rho_{\star}^2} \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \\ + \frac{n^2 T}{\rho_{\star}^2 \sigma |\log \varepsilon|^{1/2}} \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}}.$$

For any test function  $\phi \in C^2([0, T] \times \overline{\Omega})$  with compact support in  $\Omega$  and any  $0 \leq t_1 \leq t_2 \leq T$  we have

$$(6.10) \quad \int_{t_1}^{t_2} \int_{\Omega} \partial_t \left( \phi \frac{e_{\varepsilon}(u)}{|\log \varepsilon|} \right) - \int_{t_1}^{t_2} \int_{\Omega} (\partial_t \phi) \frac{e_{\varepsilon}(u)}{|\log \varepsilon|} \\ = -\frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \phi \frac{|\partial_t u|^2}{|\log \varepsilon|} - \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \nabla \phi \cdot (\partial_t u, \nabla u),$$

as is easily seen by multiplying (2.11) by  $\frac{1}{|\log \varepsilon|}$  and integrating by parts.

We now follow [25] and set

$$\phi(t, x) = \sum_{j=1}^n \phi_j(t, x) = \sum_{j=1}^n \chi(x - a_j(t)) b_j(t) \cdot (x - a_j(t)).$$

Then we calculate, dropping the  $t$ -dependence of  $a$  and  $b$ ,

$$\begin{aligned} \nabla \phi(t, x) &= \sum_{j=1}^n (b_j \cdot (x - a_j)) \nabla \chi(x - a_j) + b_j \chi(x - a_j) \\ \partial_t \phi(t, x) &= \sum_{j=1}^n -\dot{a}_j \cdot \nabla \chi(x - a_j) b_j \cdot (x - a_j) \\ &\quad + \chi(x - a_j) (\dot{b}_j \cdot (x - a_j) - b_j \cdot \dot{a}_j) \\ \nabla \partial_t \phi(t, x) &= \sum_{j=1}^n -\dot{a}_j \cdot \nabla^2 \chi(x - a_j) b_j \cdot (x - a_j) - \dot{a}_j \cdot \nabla \chi(x - a_j) b_j \\ &\quad + \nabla \chi(x - a_j) (\dot{b}_j \cdot (x - a_j) - b_j \cdot \dot{a}_j) + \chi(x - a_j) \dot{b}_j. \end{aligned}$$

In particular we have (with constants depending on  $a$  and  $b$ )

$$\begin{aligned} \|\nabla \phi\|_{L_T^\infty L_\Omega^\infty} &\lesssim \frac{1}{\sigma} \sup_j \|b_j\|_{L_T^\infty L_\Omega^\infty} \\ \|\nabla \partial_t \phi\|_{L_T^\infty L_\Omega^\infty} &\lesssim \frac{1}{\sigma^2} \sup_j \|\dot{a}_j\|_{L_T^\infty L_\Omega^\infty} \sup_j \|b_j\|_{L_T^\infty L_\Omega^\infty} + \frac{1}{\sigma} \sup_j \|\dot{b}_j\|_{L_T^\infty L_\Omega^\infty}. \end{aligned}$$

Now we analyze the terms in (6.10) one by one. We have by (3.6)

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \int_\Omega \partial_t \left( \phi \frac{e_\varepsilon(u)}{|\log \varepsilon|} \right) - \left[ \int_\Omega \phi(t_2, \cdot) \left( \pi \sum \delta_{\xi_j(t_2)} \right) - \phi(t_1, \cdot) \left( \pi \sum \delta_{\xi_j(t_1)} \right) \right] \right| \\ &\leq \left| \int_\Omega \phi(t_2, \cdot) \left( \frac{e_\varepsilon(u(t_2, \cdot))}{|\log \varepsilon|} - \pi \sum \delta_{\xi_j(t_2)} \right) \right| \\ &\quad + \left| \int_\Omega \phi(t_1, \cdot) \left( \frac{e_\varepsilon(u(t_1, \cdot))}{|\log \varepsilon|} - \pi \sum \delta_{\xi_j(t_1)} \right) \right| \\ &\lesssim \|\nabla \phi\|_{L_T^\infty L_\Omega^\infty} \sup_{t \in [t_1, t_2]} \left\| \frac{e_\varepsilon(u(t))}{|\log \varepsilon|} - \pi \sum_{j=1}^n \delta_{\xi_j(t)} \right\|_{\dot{W}^{-1,1}} \\ &\lesssim \frac{t_\varepsilon}{\sigma} \sup_j \|b_j\|_{L_T^\infty L_\Omega^\infty}. \end{aligned}$$

On the other hand

$$\begin{aligned} &\left| \sum_{j=1}^n b_j(t_2) \cdot (\xi_j(t_2) - a_j(t_2)) - b_j(t_1) \cdot (\xi_j(t_1) - a_j(t_1)) \right| \\ &\lesssim \sup_j \|b_j\|_{L_T^\infty L_\Omega^\infty} \sup_{t \in [t_1, t_2]} \sum_{j=1}^n |\xi_j(t) - a_j(t)|. \end{aligned}$$

Therefore,

$$(6.11) \quad \left| \int_{t_1}^{t_2} \int_{\Omega} \partial_t \left( \phi \frac{e_{\varepsilon}(u)}{|\log \varepsilon|} \right) \right| \lesssim \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \left[ \frac{t_{\varepsilon}}{\sigma} + \sum_{j=1}^n |\xi_j(t) - a_j(t)| \right].$$

For the second term on the left-hand side of (6.10),

$$\begin{aligned} & \left| \int_{t_1}^{t_2} \int_{\Omega} \partial_t \phi \frac{e_{\varepsilon}(u)}{|\log \varepsilon|} - \int_{t_1}^{t_2} \int_{\Omega} \partial_t \phi \sum_{j=1}^n \pi \delta_{\xi(t)} \right| \\ & \leq T \|\nabla \partial_t \phi\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_{t \in [t_1, t_2]} \left\| \frac{e_{\varepsilon}(u(t, \cdot))}{|\log \varepsilon|} - \sum_{j=1}^n \pi \delta_{\xi(t)} \right\|_{\dot{W}^{-1,1}} \\ & \lesssim t_{\varepsilon} T \left[ \frac{1}{\sigma^2} \sup_j \|\dot{a}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} + \frac{1}{\sigma} \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \right] \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{t_1}^{t_2} \int_{\Omega} \partial_t \phi \sum_{j=1}^n \pi \delta_{\xi_j} + \pi \sum_{j=1}^n \int_{t_1}^{t_2} b_j \cdot a_j \right| = \pi \left| \sum_{j=1}^n \int_{t_1}^{t_2} \dot{b}_j \cdot (\xi_j - a_j) \right| \\ & \lesssim T \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_{t \in [t_1, t_2]} \sum_{j=1}^n |\xi_j(t) - a_j(t)|. \end{aligned}$$

Thus

$$\begin{aligned} & \left| \int_{t_1}^{t_2} \int_{\Omega} \partial_t \phi \frac{e_{\varepsilon}(u)}{|\log \varepsilon|} + \pi \sum_{j=1}^n \int_{t_1}^{t_2} b_j \cdot a_j \right| \\ (6.12) \quad & \lesssim t_{\varepsilon} \frac{T}{\sigma} \left[ \frac{1}{\sigma} \sup_j \|\dot{a}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} + \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \right] \\ & \quad + T \sup_j \|\dot{b}_j\|_{L_T^{\infty} L_{\Omega}^{\infty}} \sup_{t \in [t_1, t_2]} \sum_{j=1}^n |\xi_j(t) - a_j(t)|. \end{aligned}$$

Note that the previous equality contains the term we want to appear on the left-hand side of (6.9).

For the first term on the right-hand side of (6.10) we use (6.7) and get

$$(6.13) \quad \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \phi \frac{|\partial_t u|^2}{|\log \varepsilon|} \lesssim \frac{1}{|\log \varepsilon|} \|b\|_{L_T^{\infty} L_{\Omega}^{\infty}} \frac{n^2}{\rho_{\star}}.$$

Finally, for the second term on the right-hand side of (6.10) we have

$$\begin{aligned}
 (6.14) \quad & \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \nabla \phi \cdot (\partial_t u, \nabla u) \\
 &= \sum_{j=1}^n \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{B_{2r}(a_j(t))} \nabla \chi(x - a_j) \cdot (\partial_t u, \nabla u) b_j \cdot (x - a_j) \\
 &\quad + \sum_{j=1}^n \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \chi(x - a_j) b_j \cdot (\partial_t u, \nabla u).
 \end{aligned}$$

The second term on the right-hand side of (6.14) is what we want for (6.9). We estimate the other term. Since

$$\begin{aligned}
 & \sum_{j=1}^n \frac{1}{|\log \varepsilon|} \left| \int_{t_1}^{t_2} \int_{B_{2r}(a_j(t))} \nabla \chi(x - a_j) \cdot (\partial_t u, \nabla u) b_j \cdot (x - a_j) \right| \\
 & \leq \frac{1}{|\log \varepsilon|^{1/2}} \sup_j \|\dot{b}_j\|_{L_T^\infty L_\Omega^\infty} \sup_j \|\nabla \chi(x - a_j)\|_{L_T^\infty L_\Omega^\infty} \\
 & \quad \sum_{j=1}^n \left( \int_{t_1}^{t_2} \int_{B_{2r}(a_j(t)) \setminus B_r(a_j(t))} \frac{|\partial_t u|^2}{|\log \varepsilon|} \right)^{1/2} \left( \int_{t_1}^{t_2} \int_{B_{2r}(a_j(t)) \setminus B_r(a_j(t))} |\nabla u|^2 \right)^{1/2} \\
 & \lesssim \frac{1}{\sigma |\log \varepsilon|^{1/2}} \sup_j \|\dot{b}_j\|_{L_T^\infty L_\Omega^\infty} \left[ \int_{t_1}^{t_2} \int_{\Omega} \frac{|\partial_t u|^2}{|\log \varepsilon|} + \int_{t_1}^{t_2} \int_{\Omega_\sigma(a_j(t))} |\nabla u|^2 \right] \\
 & \lesssim \frac{1}{\sigma |\log \varepsilon|^{1/2}} \sup_j \|\dot{b}_j\|_{L_T^\infty L_\Omega^\infty} \left[ \frac{n^2}{\rho_\star^2} + \frac{n^2}{\rho_\star^2} |t_2 - t_1| \right] \\
 & \lesssim \frac{n^2 |t_2 - t_1|}{\rho_\star^2 \sigma |\log \varepsilon|^{1/2}} \sup_j \|\dot{b}_j\|_{L_T^\infty L_\Omega^\infty}.
 \end{aligned}$$

3. We now study the momentum term on the left hand side of (6.9). From Cauchy-Schwarz

$$\begin{aligned}
 & \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \sum_{j=1}^n \chi(x - a_j) b_j \cdot (\partial_t u, \nabla u) \\
 & \leq \left( \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \chi(x - a) |\partial_t u|^2 \right)^{1/2} \left( \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \chi(x - a) (b \otimes b) : (\nabla u \otimes \nabla u) \right)^{1/2},
 \end{aligned}$$

where  $(b \otimes b)_{ij} = b_i b_j$  for  $b \in \mathbb{R}^2$  and  $(\nabla u \otimes \nabla u)_{ij} = (\partial_i u, \partial_j u)$  for  $u \in \mathbb{C}$ . For any  $b_j \in \mathbb{R}^2$  and  $\chi$  as above, we claim that

$$\begin{aligned}
 (6.15) \quad & \left| \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_{\Omega} \sum_{j=1}^n \chi(x - a_j) (b_j \otimes b_j) : (\nabla u \otimes \nabla u) - \int_{t_1}^{t_2} \sum_{j=1}^n \pi |b_j|^2 \right| \\
 & \lesssim \frac{nT}{|\log \varepsilon|^{1/2}} \sup_j \|b_j\|_{L_T^\infty L_\Omega^\infty}^2.
 \end{aligned}$$

In particular for any time  $t_1 \leq t \leq t_2$  we find:

$$\begin{aligned}
& \left| \frac{1}{|\log \varepsilon|} \int_{\Omega} \sum_{j=1}^n \chi(x - a_j) b_j \otimes b_j : \nabla u \otimes \nabla u - \sum_{j=1}^n \pi |b_j|^2 \right| \\
& \leq \left| \int_{\Omega} \sum_{j=1}^n \chi(x - a_j) b_j \otimes b_j : \frac{\nabla u \otimes \nabla u}{|\log \varepsilon|} \right. \\
& \quad \left. - \sum_{j=1}^n \int_{B_{\sigma}(\xi_j(t))} |b_j^x|^2 \frac{|\partial_x u|^2}{\log \frac{\sigma}{\varepsilon}} + |b_j^y|^2 \frac{|\partial_y u|^2}{\log \frac{\sigma}{\varepsilon}} \right| \\
& \quad + \sum_{j=1}^n \left| \left( \int_{B_{\sigma}(\xi_j(t))} |b_j^x|^2 \frac{|\partial_x u|^2}{\log \frac{\sigma}{\varepsilon}} + |b_j^y|^2 \frac{|\partial_y u|^2}{\log \frac{\sigma}{\varepsilon}} \right) - \pi |b_j|^2 \right| \\
& = I_1 + I_2.
\end{aligned}$$

First we analyze  $I_1$ . From (6.4) and choosing  $\sigma = \rho_{\star} \gg \varepsilon$ ,

$$\begin{aligned}
I_1 & \leq \frac{2}{|\log \varepsilon|} \sum_{j=1}^n \int_{B_{2\sigma}(a_j)} \chi(x - a_j) \left| b_j^x b_j^y (\partial_x u, \partial_y u) \right| \\
& \quad + \sum_{j=1}^n \int_{B_{2\sigma}(a_j) \setminus B_{\sigma}(\xi_j)} \chi(x - a_j) \left[ \frac{|b_j^x|^2 |\partial_x u|^2}{|\log \varepsilon|} + \frac{|b_j^y|^2 |\partial_y u|^2}{|\log \varepsilon|} \right] \\
& \quad + \left( 1 - \frac{\log \frac{\sigma}{\varepsilon}}{\log \frac{1}{\varepsilon}} \right) \sum_{j=1}^n \int_{B_{\sigma}(\xi_j)} \left[ \frac{|b_j^x|^2 |\partial_x u|^2}{|\log \varepsilon|} + \frac{|b_j^y|^2 |\partial_y u|^2}{|\log \varepsilon|} \right] \\
& \lesssim \frac{n}{|\log \varepsilon|} \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}}^2 \sqrt{\log \frac{\sigma}{\varepsilon}} \\
& \quad + \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}}^2 \int_{t_1}^{t_2} \int_{\Omega_{\sigma}(a(t))} \frac{|\nabla u|^2}{|\log \varepsilon|} \\
& \quad + \frac{\log \frac{1}{\sigma}}{|\log \varepsilon|} \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}}^2 \sum_{j=1}^n \int_{B_{\sigma}(\xi_j)} \frac{|\nabla u|^2}{\log \frac{\sigma}{\varepsilon}} \\
& \lesssim \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}}^2 \left[ \frac{n}{|\log \varepsilon|^{1/2}} + \frac{n^2}{\rho_{\star}^2 |\log \varepsilon|} + \frac{n \log \frac{1}{\sigma}}{|\log \varepsilon|} \right].
\end{aligned}$$

Next we look at  $I_2$ . Again from (6.4) and since  $\sigma \gg \varepsilon$ ,

$$I_2 \lesssim \sum_{j=1}^n |b_j|^2 \left( \log \frac{\sigma}{\varepsilon} \right)^{-1/2} \lesssim \frac{n}{|\log \varepsilon|^{1/2}} \sup_j \|b_j\|_{L_T^{\infty} L_{\Omega}^{\infty}}^2.$$

Comparing the terms from  $I_1$  and  $I_2$  yields (6.15).

4. We now use (6.9) and (6.15) to derive the expected kinetic energy estimate. Setting  $b_j = \dot{a}_j$ , noting that  $\dot{a}_j = -\frac{1}{\pi} \nabla_{a_j} W$  and  $\dot{b}_j = \ddot{a}_j = \frac{1}{\pi^2} \nabla W \nabla^2 W$ , then (2.5)

implies

$$(6.16) \quad \sup_j \|\dot{a}_j\|_{L_T^\infty L_\Omega^\infty} \lesssim \frac{n}{\rho_\star}$$

$$(6.17) \quad \sup_j \|\ddot{a}_j\|_{L_T^\infty L_\Omega^\infty} \lesssim \frac{n^2}{\rho_\star^3}.$$

So using (6.9) and (6.15) we have

$$\left| \pi \int_{t_1}^{t_2} |\dot{a}|^2 - F_1 \right| \leq \left( \int_{t_1}^{t_2} \int_\Omega \chi(x-a) \frac{|\partial_t u|^2}{|\log \varepsilon|} \right)^{1/2} \left( \pi \int_{t_1}^{t_2} |b|^2 + F_2 \right)^{1/2}.$$

where

$$F_1 = C \frac{n^4 T}{\rho_\star^6 |\log \varepsilon|^{1/2}} + C \frac{n^2 T}{\rho_\star^3} \sup_{s \in [t_1, t_2]} \sum_j^n |\xi_j(s) - a_j(s)|$$

$$F_2 = C \frac{n^3 T}{\rho_\star^2 |\log \varepsilon|^{1/2}}.$$

Square the previous inequality, obtaining by division

$$\frac{\left( \pi \int_{t_1}^{t_2} |\dot{a}|^2 - F_1 \right)^2}{\pi \int_{t_1}^{t_2} |\dot{a}|^2 + F_2} \leq \frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_\Omega \chi(x-a) |\partial_t u|^2.$$

Setting  $K = \pi \int_{t_1}^{t_2} |\dot{a}|^2$  and assuming  $|\dot{a}|^2 \geq C > 0$ , we have using

$$\frac{(K - F_1)^2}{K + F_2} = K + F_2 - 2(F_1 + F_2) + \frac{(F_1 + F_2)^2}{K + F_2}$$

that actually (dropping the final term in the previous equality)

$$\frac{1}{|\log \varepsilon|} \int_{t_1}^{t_2} \int_\Omega \chi(x-a) |\partial_t u|^2 \geq \pi \int_{t_1}^{t_2} |\dot{a}|^2 - CF_1 - CF_2,$$

and so (6.1) follows, since  $F_1 \gtrsim F_2$ .

5. We next will show that  $u(t)$  is well-approximated in certain ways by the canonical harmonic map  $u_\star(t) := u_\star(\cdot; \xi(t), d)$  for  $t \leq \tau_1$ . To do this, we need to estimate the surplus energy  $D(\xi(t))$  with respect to the points  $\xi(t)$  found in Step 1 above.

Fix  $t \in [0, \tau_1]$  and assuming  $D(a(0)) \leq \mathcal{B}_\varepsilon$  then

$$\begin{aligned} D(\xi(t)) &= D(a(t)) + W(a(t), d) - W(\xi(t), d) \\ &\leq C \mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + C \mathcal{B}_\varepsilon \\ &\quad + \left( \sum_{j=1}^n |\xi_j(t) - a_j(t)| \right) \left( \sup_j \sup_{|y-a(t)| \leq |\xi(t)-a(t)|} |\nabla_{y_j} W(y)| \right). \end{aligned}$$

where

$$\mathcal{A}_\varepsilon = \frac{n^2 T}{\rho_\star^3} \quad \mathcal{B}_\varepsilon = \frac{n^4 T}{\rho_\star^6 |\log \varepsilon|^{\frac{1}{2}}}.$$



From if  $y \in \Omega^n$  is such that  $|y - a(t)| \leq |\xi(t) - a(t)|$ , then  $\rho_y \geq \frac{1}{2}\rho_{a(t)}$  for all sufficiently small  $\varepsilon$ , and so we get

$$(6.18) \quad \begin{aligned} D(\xi(t)) &\leq C\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + C\mathcal{B}_\varepsilon + (Ct_\varepsilon + \eta(t)) \frac{Cn}{\rho_\star} \\ &\leq C\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + C\mathcal{B}_\varepsilon. \end{aligned}$$

Thus the first conclusion

$$(6.19) \quad \begin{aligned} \int_{\Omega_{\rho_\star}(\xi(t))} e_\varepsilon(|u(t)|) + \frac{1}{4} \left| \frac{j(u(t))}{|u(t)|} - j(u_\star(t)) \right|^2 \\ \leq C\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + C\mathcal{B}_\varepsilon + C \left( \frac{n^5}{\rho_\star} (s_\varepsilon + \varepsilon E_\varepsilon(u)) \right)^{\frac{1}{2}} \\ \leq C\mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) + C\mathcal{B}_\varepsilon \end{aligned}$$

for all  $t \in [0, \tau_1]$ , where (since  $s_\varepsilon \geq C\varepsilon E_\varepsilon(u)$ , see the definition (3.7)) (Note that a condition  $\sigma^* \leq \rho_a$  appearing as a hypothesis for conclusion as a result of the definitions of  $\tau_1, s_\varepsilon$  etc.)

From the other conclusion we deduce that

$$(6.20) \quad \|j(u)(t) - j(u_\star)(t)\|_{L^{4/3}(\Omega)} \leq C \left( \mathcal{A}_\varepsilon \sup_{s \in [0, t]} \eta(s) \right)^{\frac{1}{2}} + C\sqrt{\mathcal{B}_\varepsilon}.$$

□

## 7. QUANTITATIVE BOUNDS ON THE SUPERCURRENT

In this section we prove estimates of  $T_1 - T_7$  after averaging in time. As in [23] we cannot rely on the Gamma convergence estimates to bound these terms since the square root in  $\eta$  pops out. Following [23] we establish direct bounds on  $j(u) - j(u_\star)$  via a Hodge decomposition and use a time-averaging. Unfortunately, since the argument in [23] relies on very different assumptions, we need to reprove several bounds with the worse behavior in the parabolic problem. Our primary result in this section is the following bound on all  $T_j$ 's:

**Proposition 7.1.** *For all  $t \in [\delta_\varepsilon, \tau_1]$  and  $j = \{1, \dots, 7\}$*

$$(7.1) \quad |\langle T_j \rangle_{\delta_\varepsilon}(t)| \lesssim \mathcal{A}_\varepsilon \sup_{s \in [\delta, t]} \langle \eta \rangle_{\delta_\varepsilon}(s) + |\log \varepsilon|^{-\frac{11}{40}},$$

where  $\mathcal{A}_\varepsilon = C \frac{n^2 T}{\rho_\star^4}$  which satisfy (2.12), (2.15), (2.14).

Terms  $T_1$  through  $T_4$  can be directly estimated from the non-averaged estimates in Section 4. However, we are unable to bound the rest of the terms without resorting to explicit time-averaging estimates of the differential identity (2.10). The following lemma addresses the much harder terms.

**Lemma 7.2.** *For all  $t \in [\delta_\varepsilon, \tau_1]$  and  $j = \{5, 6, 7\}$*

$$(7.2) \quad |\langle T_j \rangle_{\delta_\varepsilon}(t)| \lesssim \mathcal{A}_\varepsilon \sup_{s \in [\delta, t]} \langle \eta \rangle_{\delta_\varepsilon}(s) + |\log \varepsilon|^{-\frac{11}{40}},$$

where  $\mathcal{A}_\varepsilon = C \frac{n^2 T}{\rho_\star^4}$  which satisfy (2.12), (2.15), (2.14).

*Proof.* For simplicity we write

$$T_5 = \int_{\Omega} \zeta_k \left( \frac{j(u)}{|u|} - j(u_\star) \right)_k$$

where

$$(7.3) \quad \zeta_k := \sum_j \mathbb{J}_{kl} \partial_{x_l x_m} \left( \frac{\eta_j}{|\eta_j|} \cdot \Phi_j \right) j_m(u_\star), \quad k = 1, 2,$$

$\mathbb{J}_{12} = -\mathbb{J}_{21} = 1$ ,  $\mathbb{J}_{11} = \mathbb{J}_{22} = 0$ , and  $j_m$  denotes the  $m$  component of  $j(u_\star)$ ,  $m = 1, 2$ . Here  $u_\star(x, t) = u_\star(x; \xi(t))$  as usual.

1. From the definitions and (2.4) one finds  $|\zeta| \leq C \frac{n}{\rho_\star^2}$ , and  $|\text{supp } \zeta| \leq C n \rho_\star^2$ . It follows that

$$(7.4) \quad \|\zeta\|_{L^q(\Omega)} \lesssim n^{1+\frac{1}{q}} \rho_\star^{\frac{2}{q}-2}$$

for  $1 \leq q \leq \infty$ .

The following proof is quite similar to the proof found in Proposition 1 in [23]; however, we include it since the bounds are different, due to a different conservation law. We perform a Hodge decomposition

$$(7.5) \quad j(u) - j(u_\star) = \nabla f_1 + \nabla^\perp f_2$$

with boundary conditions either

$$(7.6) \quad f_1 = 0 \text{ and } \partial_\nu f_2 = 0 \text{ on } \partial\Omega,$$

or

$$(7.7) \quad \partial_\nu f_1 = 0 \text{ and } f_2 = 0 \text{ on } \partial\Omega,$$

depending whether we are dealing with Dirichlet or Neumann boundary conditions. And so we examine

$$\begin{aligned} \Delta f_1 &= \text{div } j(u) \\ -\Delta f_2 &= 2 \left[ J(u) - \sum \pi d_j \delta_{\xi_j} \right] \end{aligned}$$

with (7.6) or (7.7).

Since  $\nabla f_1$  is small only after time-averaging, we write our estimate as

$$\begin{aligned}
\langle T_5 \rangle_{\delta_\varepsilon} &= \left\langle \int \zeta \cdot \frac{j(u)}{|u|} (1 - |u|) \right\rangle_{\delta_\varepsilon} + \left\langle \int \zeta \cdot \operatorname{curl} f_2 \right\rangle_{\delta_\varepsilon} + \left\langle \int \zeta \cdot \nabla f_1 \right\rangle_{\delta_\varepsilon} \\
&= \left\langle \int \zeta \cdot \frac{j(u)}{|u|} (1 - |u|) \right\rangle_{\delta_\varepsilon} + \left\langle \int \zeta \cdot \operatorname{curl} f_2 \right\rangle_{\delta_\varepsilon} + \left\langle \int \langle \zeta \rangle_{\delta_\varepsilon} \cdot \langle \nabla f_1 \rangle_{\delta_\varepsilon} \right\rangle_{\delta_\varepsilon} \\
&\quad + \left\langle \int (\zeta - \langle \zeta \rangle_{\delta_\varepsilon}) \cdot (\nabla f_1 - \langle \nabla f_1 \rangle_{\delta_\varepsilon}) \right\rangle_{\delta_\varepsilon} \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

The first term is estimated by the Cauchy-Schwarz inequality,

$$(7.8) \quad |A_1| \leq \|\zeta\|_{L^\infty} \left\| \frac{j(u)}{|u|} \right\|_{L^2} \|(1 - |u|^2)\|_{L^2} \leq C \frac{n}{\rho_\star^2} \varepsilon E_\varepsilon(u) \lesssim |\log \varepsilon|^{-\frac{1}{3}}.$$

2. Next we claim that

$$(7.9) \quad |A_2| \leq C s_\varepsilon^{\frac{3}{5}} \left[ n^{\frac{6}{5}} \rho_\star^{-\frac{8}{5}} (E_\varepsilon(u) + n\pi)^{\frac{2}{5}} \right] \lesssim |\log \varepsilon|^{-\frac{1}{3}}.$$

From the Hodge decomposition and standard elliptic estimates [41] we have

$$\|\operatorname{curl} f_2\|_{L^p(\Omega)} \leq \|f_2\|_{W^{1,p}(\Omega)} \leq C s_\varepsilon^{\frac{2}{p}-1} (E_\varepsilon(u) + n)^{2-\frac{2}{p}}$$

for  $1 \leq p < 2$ , with a constant depending on  $p$ . Taking  $\frac{1}{q} = 1 - \frac{1}{p}$  in (7.4) for  $p \in [1, 2)$  to be selected, we conclude that

$$|A_2| \leq \|\zeta\|_{L^q} \|\operatorname{curl} f_2\|_{L^p} \leq C n^{2-\frac{1}{p}} \rho_\star^{-\frac{2}{p}} s_\varepsilon^{\frac{2}{p}-1} (E_\varepsilon(u) + n\pi)^{2-\frac{2}{p}}.$$

Choosing  $p = \frac{5}{4}$ , we arrive at (7.9).

3. Next, we estimate  $A_3$ , and here we fundamentally use the time-averaging to control  $\nabla f_1$ .

$$\begin{aligned}
\|\Delta \langle f_1 \rangle_{\delta_\varepsilon}\|_{L^2}^2 &= \|\operatorname{div} \langle j(u) - j(u_\star) \rangle_{\delta_\varepsilon}\|_{L^2}^2 = \|\langle \operatorname{div} j(u) \rangle_{\delta_\varepsilon}\|_{L^2}^2 \\
&= \left\| \frac{1}{\delta_\varepsilon} \int_{t-\delta_\varepsilon}^t \frac{(iu, \partial_t u)}{|\log \varepsilon|} \right\|_{L^2}^2 \\
&\leq \int_\Omega \frac{1}{\delta_\varepsilon |\log \varepsilon|} \int_{t-\delta_\varepsilon}^t \frac{|\partial_t u|^2}{|\log \varepsilon|} \lesssim \frac{1}{\delta_\varepsilon |\log \varepsilon|} \frac{n^2}{\rho_\star^2}.
\end{aligned}$$

By standard elliptic estimates

$$\|\langle f_1 \rangle_{\delta_\varepsilon}\|_{H^2} \lesssim \|\Delta \langle f_1 \rangle_{\delta_\varepsilon}\|_{L^2} \lesssim \frac{n}{\delta_\varepsilon^{\frac{1}{2}} \rho_\star |\log \varepsilon|^{\frac{1}{2}}}.$$

Combining with (2.12), (2.13), (2.15) yields

$$\begin{aligned}
\left| \int \langle \zeta \rangle_{\delta_\varepsilon} \cdot \langle \nabla f_1 \rangle_{\delta_\varepsilon} \right| &\leq \| \langle \zeta \rangle_{\delta_\varepsilon} \|_{L^{4/3}} \| \langle \nabla f_1 \rangle_{\delta_\varepsilon} \|_{L^4} \\
&\leq C \langle \| \zeta \|_{L^{4/3}} \rangle_{\delta_\varepsilon} \| \langle \nabla f_1 \rangle_{\delta_\varepsilon} \|_{H^1} \\
&\lesssim \frac{n^{\frac{11}{4}}}{\delta_\varepsilon^{\frac{1}{2}} \rho_\star^{\frac{3}{2}} |\log \varepsilon|^{\frac{1}{2}}} \lesssim |\log \varepsilon|^{-\frac{277}{800}};
\end{aligned}$$

hence,

$$(7.10) \quad |A_3| \lesssim |\log \varepsilon|^{-\frac{1}{3}}.$$

4. Finally, we consider the challenging term  $A_4$ , and we again following the strategy of [23]. The idea is to take advantage of the fact that  $\delta_\varepsilon$  is small to show that  $\zeta$  is close to  $\langle \zeta \rangle_{\delta_\varepsilon}$ , and similarly  $\nabla f_1$  and  $\langle \nabla f_1 \rangle_{\delta_\varepsilon}$ . First we have

$$\begin{aligned}
|A_4| &\leq \sup_{s \in [t-\delta_\varepsilon, t]} \| \zeta(s) - \langle \zeta \rangle_{\delta_\varepsilon} \|_{L^4(\Omega)} \sup_{s \in [t-\delta_\varepsilon, t]} \| \nabla(f_1(s) - \langle f_1 \rangle_{\delta_\varepsilon}) \|_{L^{\frac{4}{3}}(\Omega)} \\
(7.11) \quad &\leq \sup_{s, s' \in [t-\delta_\varepsilon, t]} \| \zeta(s) - \zeta(s') \|_{L^4(\Omega)} \sup_{s, s' \in [t-\delta_\varepsilon, t]} \| \nabla(f_1(s) - f_1(s')) \|_{L^{\frac{4}{3}}(\Omega)}.
\end{aligned}$$

In estimating the quantities in (7.11), we will use that for  $s, s' \in [t - \delta_\varepsilon, t]$  with  $t \in [\delta_\varepsilon, \tau_1]$ ,

$$(7.12) \quad |a_j(s) - a_j(s')| \lesssim \frac{n}{\rho_\star} \delta_\varepsilon,$$

which follows from (1.11). Note that (7.12) and (2.5) imply that  $|a_j| \lesssim \frac{n}{\rho_\star}$ . From (7.12) and (3.14), (3.18) it follows that for  $s, s'$  as above,

$$(7.13) \quad \sum_{j=1}^n |\xi_j(s) - \xi_j(s')| \leq C \frac{n^2}{\rho_\star} \delta_\varepsilon + \eta(s) + \eta(s') + C t_\varepsilon \lesssim \frac{n^2}{\rho_\star} \delta_\varepsilon + \mathcal{D}_\varepsilon.$$

4a. We estimate  $\| \nabla(f_1(s) - f_1(s')) \|_{L^{\frac{4}{3}}(\Omega)}$ . Assume that  $s, s' \in [t - \delta_\varepsilon, t]$  for  $t \in [\delta_\varepsilon, \tau_1]$ . By elliptic regularity, (7.5), and either (7.6) or (7.7) we find that,

$$\begin{aligned}
\| \nabla(f_1(s) - f_1(s')) \|_{L^{\frac{4}{3}}(\Omega)} &\leq C \| \Delta(f_1(s) - f_1(s')) \|_{W^{-1, \frac{4}{3}}(\Omega)} \\
&= \| \nabla \cdot [j(u)(s) - j(u)(s')] \|_{W^{-1, \frac{4}{3}}(\Omega)} \\
(7.14) \quad &\leq \| j(u)(s) - j(u)(s') \|_{L^{\frac{4}{3}}(\Omega)}.
\end{aligned}$$

Using the triangle inequality and (6.20), it follows that

$$\begin{aligned}
\| j(u)(s) - j(u)(s') \|_{L^{\frac{4}{3}}(\Omega)} &\leq C \sqrt{\mathcal{A}_\varepsilon \sup_{r \in [\delta, s]} \eta(r)} + C \sqrt{\mathcal{B}_\varepsilon} \\
&\quad + \| j(u_\star)(s) - j(u_\star)(s') \|_{L^{\frac{4}{3}}(\Omega)} \\
&\lesssim |\log \varepsilon|^{-\frac{21}{200}} |\log |\log \varepsilon||^{\frac{1}{2}} \\
&\quad + \| j(u_\star)(s) - j(u_\star)(s') \|_{L^{\frac{4}{3}}(\Omega)}.
\end{aligned}$$

The last term on the right-hand side can be estimated by combining (2.8) and (7.13), and we get

$$\begin{aligned} \|j(u_\star)(s) - j(u_\star)(s')\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim n^{\frac{1}{2}} \left( \delta_\varepsilon \frac{n^2}{\rho_\star} + \mathcal{D}_\varepsilon \right)^{\frac{1}{2}} \\ &\lesssim \frac{n^{\frac{3}{2}}}{\rho_\star^{\frac{1}{2}}} \delta_\varepsilon^{\frac{1}{2}} + n^{\frac{1}{2}} \mathcal{D}_\varepsilon^{\frac{1}{2}} \lesssim |\log \varepsilon|^{-\frac{9}{80}}. \end{aligned}$$

The rest of the terms on the right-hand side of (7.14) are smaller using the bounds on  $n, \rho_\star$ . Therefore, we find that

$$(7.15) \quad \|\nabla(f_1(s) - f_1(s'))\|_{L^{\frac{4}{3}}(\Omega)} \lesssim |\log \varepsilon|^{-\frac{21}{200}} |\log |\log \varepsilon||^{\frac{1}{2}}.$$

4b. We estimate  $\|\zeta(s) - \zeta(s')\|_{L^4(\Omega)}$ . Assume that  $0 \leq t - \delta_\varepsilon \leq s, s' \leq t \leq \tau_1$ . In order to find a time-Lipschitz bound on  $\zeta$ , we have from the definition (7.3) that

$$\begin{aligned} \zeta_k(s) - \zeta_k(s') &= \sum_j \mathbb{J}_{kl} \partial_{x_l x_m} \left( \frac{\eta_j}{|\eta_j|} \cdot \Phi_j \right)(s) j_m(u_\star)(s) \\ &\quad - \sum_j \mathbb{J}_{kl} \partial_{x_l x_m} \left( \frac{\eta_j}{|\eta_j|} \cdot \Phi_j \right)(s') j_m(u_\star)(s') \\ &= \sum_j \mathbb{J}_{kl} \partial_{x_l x_m} \left[ \frac{\eta_j}{|\eta_j|} \cdot (\Phi_j(s) - \Phi_j(s')) \right] j_m(u_\star)(s) \\ &\quad + \sum_j \mathbb{J}_{kl} \partial_{x_l x_m} \left( \frac{\eta_j}{|\eta_j|} \cdot \Phi_j \right)(s') [j_m(u_\star)(s) - j_m(u_\star)(s')] \\ &= Z_1 + Z_2. \end{aligned}$$

First consider  $Z_1$ . From the definitions,

$$\begin{aligned} \left\| \partial_{x_l x_m} \left[ \frac{\eta_j}{|\eta_j|} \cdot (\Phi_j(s) - \Phi_j(s')) \right] \right\|_{L^\infty(\Omega)} &\leq \left\| \partial_{x_l x_m} [\varphi(x - a_j(s)) - \varphi(x - a_j(s'))] \right\|_{L^\infty(\Omega)} \\ &\leq C \|\partial_{x_l x_m x_n} \varphi\|_{L^\infty} |a_j(s) - a_j(s')| \lesssim \frac{n}{\rho_\star^3} \delta_\varepsilon \end{aligned}$$

using (7.12).

As in [23] we claim that

$$(7.16) \quad \text{supp } \nabla^2 \Phi_j(s) \cup \text{supp } \nabla^2 \Phi_j(s') \subset B_{3\rho_\star}(\xi_j(s)) \setminus B_{\frac{1}{2}\rho_\star}(\xi_j(s))$$

for all  $\varepsilon$  sufficiently small. This follows from (7.12), (7.13), and (3.14). In particular the distances separating  $a_i(s), a_i(s'), \xi_i(s), \xi_i(s')$  are significantly smaller than  $|\log \varepsilon|^{-\frac{7}{200}} \leq \rho_\star$ .

The support condition (7.16) implies that  $|j(u_\star)(\xi(s))| \leq \frac{Cn}{\rho_\star}$  on the support of  $Z_1$ . Since the support of  $Z_1$  has measure bounded by  $Cn\rho_\star^2$ , we conclude that

$$(7.17) \quad \|Z_1\|_{L^4} \leq \frac{Cn^2}{\rho_\star^4} (Cn\rho_\star^2)^{\frac{1}{4}} \delta_\varepsilon \lesssim \frac{n^{\frac{9}{4}}}{\rho_\star^{\frac{7}{2}}} \delta_\varepsilon \lesssim |\log \varepsilon|^{-\frac{163}{800}}.$$

Finally we consider  $Z_2$ . Since  $\left\| \sum_j \partial_{x_l x_m} \Phi_j \right\|_{L^\infty} \lesssim \frac{1}{\rho_\star}$ , and using that  $\text{supp } Z_2$  has measure at most  $Cn\rho_\star^2$ , we use Hölder's inequality to estimate

$$\|Z_2\|_{L^4} \leq \frac{C}{\rho_\star} \|j(u_\star)(s) - j(u_\star)(s')\|_{L^\infty(\cup_j \text{supp } \nabla^2 \Phi_j(s'))} (Cn\rho_\star^2)^{\frac{1}{4}}.$$

It then follows that  $\text{supp } \cup_j \nabla^2 \Phi_j(s') \subset \Omega_{\rho_\star/2}(\xi(s)) \cap \Omega_{\rho_\star/2}(\xi(s'))$ . We therefore use (2.7) to find that

$$\|j(u_\star)(s) - j(u_\star)(s')\|_{L^\infty(\cup_j \text{supp } \nabla^2 \Phi_j(s'))} \leq \frac{C}{\rho_\star^2} \sum_{j=1}^n |\xi_j(s) - \xi_j(s')|.$$

Consequently, (7.13) and (4.9) imply that

$$\begin{aligned} \|Z_2\|_{L^4} &\leq C \left( \frac{n}{\rho_\star^2} \right)^{\frac{5}{4}} \left( \frac{n^2}{\rho_\star} \delta_\varepsilon + \eta(s) + \eta(s') + Ct_\varepsilon \right) \\ &\lesssim \frac{n^{\frac{5}{4}}}{\rho_\star^{\frac{5}{2}}} \sup_{\delta \leq s \leq t} \langle \eta(t) \rangle_{\delta_\varepsilon} + \frac{n^{\frac{5}{4}}}{\rho_\star^{\frac{5}{2}}} \left[ \frac{n^2}{\rho_\star} \delta_\varepsilon + \delta_\varepsilon \frac{n^{\frac{3}{2}}}{\rho_\star} \sqrt{\mathcal{A}_\varepsilon \mathcal{D}_\varepsilon + \mathcal{B}_\varepsilon} + t_\varepsilon \right] \\ (7.18) \quad &\lesssim \frac{n^2 T}{\rho_\star^4} \sup_{\delta \leq s \leq t} \langle \eta(t) \rangle_{\delta_\varepsilon} + |\log \varepsilon|^{-\frac{49}{160}} |\log |\log \varepsilon||^{\frac{1}{2}}. \end{aligned}$$

Combining (7.17) and (7.18) yields

$$(7.19) \quad \|\zeta(s) - \zeta(s')\|_{L^4(\Omega)} \leq C \frac{n^2 T}{\rho_\star^4} \sup \langle \eta(t) \rangle + C |\log \varepsilon|^{-\frac{163}{800}}.$$

5. Finally we combine the above with (7.11) and (7.19) to deduce that

$$|A_4| \lesssim \frac{n^2 T}{\rho_\star^4} \sup \langle \eta(t) \rangle + |\log \varepsilon|^{-\frac{3}{10}}$$

which completes the proof of Lemma 7.2.  $\square$

Finally we can finish the proof of Proposition 7.1 which entails showing that time-averaging does not damage estimates from Section 4.

*Proof of Proposition 7.1.* Since  $\langle T_5 \rangle_{\delta_\varepsilon}$  through  $\langle T_7 \rangle_{\delta_\varepsilon}$  are covered in Lemma 7.2, we concentrate on turning the  $T_1 - T_4$  estimates into a time-averaged estimate.

Since  $\eta$  is continuous, we have for some  $c \in [\delta_\varepsilon, t]$ ,

$$\begin{aligned} \frac{1}{\rho_\star} \mathcal{A}_\varepsilon \sup_{s \in [\delta_\varepsilon, t]} \eta(s) &= \frac{1}{\rho_\star} \mathcal{A}_\varepsilon \eta(c) \\ &\lesssim \frac{1}{\rho_\star} \mathcal{A}_\varepsilon \left[ \langle \eta \rangle_{\delta_\varepsilon}(c) + C |\log \varepsilon|^{-\frac{27}{80}} |\log |\log \varepsilon||^{\frac{1}{2}} \right] \\ &\lesssim \frac{n^2 T}{\rho_\star^4} \left[ \sup_{s \in [\delta_\varepsilon, t]} \langle \eta \rangle_{\delta_\varepsilon}(s) + C |\log \varepsilon|^{-\frac{27}{80}} |\log |\log \varepsilon||^{\frac{1}{2}} \right] \end{aligned}$$

and since  $\frac{n^2 T}{\rho_\star^4} |\log \varepsilon|^{-\frac{27}{80}} |\log |\log \varepsilon||^{\frac{1}{2}} \leq |\log \varepsilon|^{-\frac{11}{40}}$ , the result follows.  $\square$

## 8. COMPLETION OF GRONWALL ARGUMENT

We now complete the proof of our Theorem 1.1. The proof entails an argument that extends  $\tau_2$  up to time  $T$  which ensures that the required estimates hold.

*Proof of Theorem 1.1.* Let  $T = \min\{\sqrt{\log|\log\varepsilon|\frac{\rho_x^2}{n}}, \tau_0\}$  denote the longest possible time interval of existence. The main point of the proof will be to show that we can extend the time up to  $T$  by a combination of continuity arguments for the Jacobian and a Gronwall estimate on  $\langle\eta\rangle_{\delta_\varepsilon}$ .

1. We first claim that (1.1) is a continuous operator from  $t \mapsto \dot{H}^1$ , i.e.

$$(8.1) \quad \|\nabla u(t) - \nabla u(s)\|_{L^2} \leq C_\varepsilon \omega(|t - s|)$$

for  $0 \leq s \leq t \leq T$ , where  $C_\varepsilon$  can depend on  $\varepsilon$  and  $T$ . It is standard theory (see for instance [15] Section 5.9, Theorem 4) that if  $u \in L^2(0, T; H^2(\Omega; \mathbb{C}))$  and  $\partial_t u \in L^2(0, T; L^2(\Omega; \mathbb{C}))$  then  $u \in C^0([0, T]; H^1(\Omega; \mathbb{C}))$ , which implies (8.1). These conditions are true for solutions of (1.1) since by the gradient flow property,

$$\int_0^T \int_\Omega |\partial_t u|^2 \leq C E_\varepsilon(u(0)) \leq C_\varepsilon$$

and from (2.9) one finds  $\|u\|_{L^\infty} \leq 1$  due to the maximum principle; hence,

$$\begin{aligned} \int_0^T \int_\Omega |\Delta u|^2 &\leq C \int_0^T \int_\Omega |\partial_t u|^2 + C \int_0^T \int_\Omega \frac{1}{\varepsilon^2} |u|^2 \frac{(1 - |u|^2)^2}{\varepsilon^2} \\ &\leq C E_\varepsilon(u(0)) + CT \|u(t)\|_{L^\infty(\Omega)}^2 E_\varepsilon(u_0) \leq C_\varepsilon, \end{aligned}$$

where  $C_\varepsilon$  can depend on  $T$  and  $\varepsilon$ .

2. Since  $J(u) = \det \nabla u = -\nabla^\perp u_1 \cdot \nabla u_2$ , where  $u = u_1 + iu_2$ . Then for any  $0 \leq s \leq t \leq T$

$$\begin{aligned} \|J(u)(t) - J(u)(s)\|_{\dot{W}^{-1,1}(\Omega)} &= \sup_{\|\phi\|_{W_0^{1,\infty}(\Omega)} \leq 1} \left| \int \phi (J(u)(t) - J(u)(s)) \right| \\ &\leq C \|\nabla u(t) - \nabla u(s)\|_{L^2} \|\nabla u(t) + \nabla u(s)\|_{L^2} \\ &\leq C_\varepsilon \omega(|t - s|) \end{aligned}$$

with modulus of continuity  $\omega(s)$  from above and  $C_\varepsilon$ , a constant depending on  $\varepsilon$  and  $T$ .

3. Suppose that  $\tau_2 = \tau_1 < T$ , then we claim that we can extend  $\tau_1 \mapsto \tau_1 + \mu$  for some  $\mu > 0$ . Suppose not, then at  $t = \tau_1$  we have

$$(8.2) \quad D(a(\tau_1)) \leq 100$$

and

$$(8.3) \quad \left\| J(u)(\tau_1) - \sum_{j=1}^n d_j \delta_{a_j(\tau_1)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leq C_\star |\log \varepsilon|^{-\frac{7}{100}},$$

where  $C_\star$  is defined in Proposition 5.2.

Consider first (8.2). Since  $\tau_2 = \tau_1$  then  $\eta(\tau_1) \leq \mathcal{D}_\varepsilon \lesssim |\log \varepsilon|^{-\frac{1}{5}}$ . From (6.3) we see that in fact  $D(a(\tau_1)) \ll 1 \ll 100$ . We now claim that there exists a  $\mu_0$  such that for all  $\tau_1 \leq \tilde{t} \leq \tau_1 + \mu_0$  we have  $D(a(\tilde{t})) \leq 100$ . In particular, by (1.11) and (2.5)

$$\begin{aligned} D(a(\tilde{t})) &= D(a(\tau_1)) + \int_{\tau_1}^{\tilde{t}} |\dot{a}|^2 - \int_{\tau_1}^{\tilde{t}} \int_{\Omega} \frac{|\partial_t u|^2}{\pi |\log \varepsilon|} \\ &\leq D(a(\tau_1)) + \int_{\tau_1}^{\tilde{t}} |\dot{a}|^2 \\ &\leq D(a(\tau_1)) + C\mu_0 \frac{n^2}{\rho_\star^2} \leq 100 \end{aligned}$$

for  $\mu_0$  small enough.

Next consider (8.3). Again  $\eta(\tau_1) \lesssim |\log \varepsilon|^{-\frac{1}{5}}$  and so by Lemma 3.2 we have

$$\|J(u)(\tau_1) - \sum d_j \delta_{a_j(\tau_1)}\|_{\dot{W}^{-1,1}} = C_\star |\log \varepsilon|^{-\frac{1}{5}} \leq \frac{C_\star}{3} |\log \varepsilon|^{-\frac{7}{200}}.$$

By Step 2 there exists a  $\mu_1 > 0$  such that for all  $\tau_1 \leq \tilde{t} \leq \tau_1 + \mu_1$

$$\|J(u)(\tilde{t}) - J(u)(\tau_1)\|_{\dot{W}^{-1,1}} \leq C(E_\varepsilon(u_0))\omega(\mu) \leq \frac{C_\star}{3} |\log \varepsilon|^{-\frac{7}{200}}$$

for  $\mu_1$  small enough. Furthermore, there exists  $\mu_2$  such that for  $\tau_1 \leq \tilde{t} \leq \tau_1 + \mu_2$

$$\begin{aligned} \|\pi \sum \delta_{a_j(\tilde{t})} - \pi \sum \delta_{a_j(\tau_1)}\|_{\dot{W}^{-1,1}} &\leq C \sum |a_j(\tilde{t}) - a_j(\tau_1)| \leq C\mu_2 \sum |\nabla_{a_j} W| \\ &\leq C \frac{n^2}{\rho_\star} \mu_2 \leq \frac{C_\star}{3} |\log \varepsilon|^{-\frac{7}{200}} \end{aligned}$$

for  $\mu_2$  small enough, where we used (2.5) in the third inequality. Therefore, for  $\mu = \min\{\mu_0, \mu_1, \mu_2\} > 0$  and all  $\tau_1 \leq \tilde{t} \leq \tau_1 + \mu$  we have

$$\begin{aligned} \left\| J(u)(\tilde{t}) - \sum \pi \delta_{a_j(\tilde{t})} \right\|_{\dot{W}^{-1,1}} &\leq \|J(u)(\tilde{t}) - J(u)(\tau_1)\|_{\dot{W}^{-1,1}} \\ &\quad + \|\pi \sum \delta_{a_j(\tilde{t})} - \pi \sum \delta_{a_j(\tau_1)}\|_{\dot{W}^{-1,1}} \\ &\quad + \|J(u)(\tau_1) - \sum d_j \delta_{a_j(\tau_1)}\|_{\dot{W}^{-1,1}} \\ &\leq C_\star |\log \varepsilon|^{-\frac{7}{200}}, \end{aligned}$$

and  $D(a(\tilde{t})) \leq 100$ . Hence, we can extend  $\tau_1$  to  $\tau_1 + \mu$ .

4. Assume that  $T \geq \delta_\varepsilon \geq \tau_1 > \tau_2$ , then we claim that  $\tau_1 \geq \tau_2 \geq \delta_\varepsilon$ . Since  $\tau_2 > 0$  we can use (4.8).

$$|\dot{\eta}| \lesssim |\log \varepsilon|^{-\frac{7}{80}} |\log |\log \varepsilon||^{\frac{1}{2}}$$

then

$$\eta(t) \leq \eta(0) + tC |\log \varepsilon|^{-\frac{7}{80}} |\log |\log \varepsilon||^{\frac{1}{2}} \leq \frac{1}{2} \mathcal{D}_\varepsilon.$$

Hence for all  $0 \leq t \leq |\log \varepsilon|^{-\frac{1}{6}}$  then  $\eta(t) \leq \frac{1}{2} \mathcal{D}_\varepsilon$ . In particular, this shows that  $\tau_2 = \tau_1$  up to  $\delta_\varepsilon \leq |\log \varepsilon|^{-\frac{1}{6}}$ . From Step 3 we can extend  $\tau_1$  and  $\tau_2$  to at least  $\delta_\varepsilon$ .



5. Assume that  $T \geq \tau_1 > \tau_2 \geq \delta_\varepsilon$ , then we claim that  $\tau_1 \geq \tau_2 \geq T$ . Note that  $\eta_3 \leq \eta_2$  using (4.9). Note that

$$\langle \eta(\delta_\varepsilon) \rangle_{\delta_\varepsilon} \leq \eta(0) + C\delta_\varepsilon |\log \varepsilon|^{-\frac{7}{80}} |\log |\log \varepsilon||^{\frac{1}{2}} \leq \frac{1}{8} \mathcal{D}_\varepsilon.$$

From Proposition 7.1 we have the differential inequality for the averaged  $\langle \eta \rangle_{\delta_\varepsilon}$ :

$$\frac{d}{dt} \langle \eta \rangle_{\delta_\varepsilon} \leq |\langle T_j \rangle_{\delta_\varepsilon}(t)| \lesssim \frac{n^2 T}{\rho_\star^4} \sup_{s \in [\delta_\varepsilon, t]} \langle \eta \rangle_{\delta_\varepsilon}(s) + |\log \varepsilon|^{-\frac{11}{40}}.$$

Using the Gronwall argument from Lemma 8.1 below, we find

$$\langle \eta(t) \rangle_{\delta_\varepsilon} \leq \left( \langle \eta(\delta_\varepsilon) \rangle_{\delta_\varepsilon} + |\log \varepsilon|^{-\frac{11}{40}} \frac{\rho_\star^4}{n^2 T} \right) \exp \left[ \frac{n^2 T^2}{\rho_\star^4} \right] \leq \frac{1}{4} \mathcal{D}_\varepsilon.$$

for  $\varepsilon \leq \varepsilon_0$  for all  $\delta_\varepsilon \leq t \leq T = \min\{\frac{\rho_\star^2}{n} \sqrt{\log |\log \varepsilon|}, \tau_0\}$ . By (4.9) we find  $\eta(t) \leq \frac{1}{2} \mathcal{D}_\varepsilon$  for  $t$  in the same time interval.  $\square$

We conclude with the following Gronwall estimate used at the end of the proof of Theorem 1.1.

**Lemma 8.1.** *Suppose  $A, B$  are positive constants and suppose*

$$\frac{d}{dt} x(t) \leq A \sup_{s \in [0, t]} x(s) + B$$

*then*

$$(8.4) \quad x(t) \leq \left[ x_0 + \frac{B}{A} \right] e^{At}.$$

*Proof.* Let  $m(t) = \sup_{s \in [0, t]} x(s)$  then  $\dot{m}(t) = \max\{\dot{x}(t), 0\}$  since the maximum can increase only if  $x$  increases. On the one hand, if  $\dot{m}(t) = \dot{x}(t)$  then  $\dot{m}(t) = \dot{x}(t) \leq Am(t) + B$ . On the other hand, if  $\dot{m}(t) = 0$  then  $\dot{m}(t) \leq Am(t) + B$ . The estimate follows.  $\square$

## 9. ESTIMATES ON THE NEUMANN FUNCTION AND THE RENORMALIZED ENERGY

In this section we provide estimates on the Neumann functions that comprise the renormalized energy in the Dirichlet case. These estimates will be used both to generate long-lived solutions of (1.1) with asymptotically many vortices and to provide kernel estimates for the hydrodynamic limit in the next section. The challenge that we face is to control how close the  $a_j(t)$  can get to each other or to the boundary using solely information on the renormalized energy  $W(a)$ . This provides only exponentially weak control and leads to dilute concentrations for asymptotically long times.

The following proposition gives a class of data for which we can be assured that the point masses stay separated enough to use Theorem 1.1 on a long enough time scale. In particular we need  $T \gtrsim \frac{1}{n}$ .

**Proposition 9.1.** *Fix  $\varepsilon$  and suppose  $u_\varepsilon(0)$  is initial data satisfying the hypotheses of Theorem 1.1 in the Dirichlet case with  $u_\varepsilon(0) = e^{in\theta + i\varphi_\star}$  for  $\varphi_\star \in C^2$  and  $\int_{\partial\Omega} \varphi_\star = 0$ . Furthermore, assume  $\rho_{a(0)} \geq |\log |\log |\log \varepsilon|||^{-\frac{1}{3}}$  and  $n \leq |\log |\log |\log \varepsilon|||^{-\frac{1}{4}}$  then solution  $u_\varepsilon(t)$  has vortices close to the  $a_j(t)$ 's (in the sense of Theorem 1.1), which satisfy*

$$\rho_{a(t)} \geq |\log |\log \varepsilon|||^{-\frac{1}{10}}$$

for all  $0 \leq t \leq |\log |\log \varepsilon|||^{-\frac{1}{7}}$ .

In order to prove Proposition 9.1 we follow the approach of Sandier-Soret [45] to define the renormalized energy in terms of Neumann functions. In particular let  $N_n(x, y)$  denote the Neumann function which satisfies the following equation

$$(9.1) \quad \begin{aligned} \Delta N_n(\cdot, y) &= \delta_y \text{ in } \Omega \\ \partial_\nu N_n(\cdot, y) &= \partial_\tau \theta + \frac{1}{n} \partial_\tau \varphi_\star \text{ on } \partial\Omega. \end{aligned}$$

and the limiting Neumann function  $N(x, y) = N_\infty(x, y)$  which satisfies the following equation

$$(9.2) \quad \begin{aligned} \Delta N(\cdot, y) &= \delta_y \text{ in } \Omega \\ \partial_\nu N(\cdot, y) &= \partial_\tau \theta \text{ on } \partial\Omega. \end{aligned}$$

We also define  $H_n(x, y) = N_n(x, y) - \log |x - y|$  and  $H(x, y) = N(x, y) - \log |x - y|$  to be the harmonic pieces of the Neumann functions  $N_n(x, y)$  and  $N(x, y)$ , respectively. Then

$$(9.3) \quad W(a_1, \dots, a_n) = -\pi \sum_{j \neq k} N_n(a_j, a_k) - \pi \sum_{j=1}^n H_n(a_j, a_j),$$

see [3] for example.

We state the following useful set of estimates:

**Lemma 9.2** (Sandier-Soret [45]). *The Neumann function  $N_n(x, y)$  verifies*

- (1)  $N_n(x, y) = N_n(y, x)$
- (2)  $N_n(x, y) = \log |x - y| + H_n(x, y)$  where  $H_n(x, y)$  is continuous on  $\Omega \times \overline{\Omega} \cup \overline{\Omega} \times \Omega$ .
- (3)  $N_n(x, y) = 2 \log |x - y| + \tilde{H}_n(x, y)$  where  $\tilde{H}_n$  is continuous on  $\partial\Omega \times \overline{\Omega} \cup \overline{\Omega} \times \partial\Omega$ .

for all  $1 \leq n \leq \infty$ .

In the proof of Lemma 9.2 the authors generate  $H_n(x, y)$  in steps. When  $\Omega = B_1$  and  $\partial_\nu N(\cdot, y) = 1$  then  $H(x, y) = \hat{H}(x, y)$ , where the complexified  $\hat{H}$  is explicit:

$$(9.4) \quad \hat{H}(x, y) = \log |1 - x\bar{y}|.$$

For nontrivial  $\partial_\nu N_n(\cdot, y) = f_n = \partial_\tau \theta + \frac{1}{n} \varphi_\star$  one finds  $\hat{H}_n$  satisfies  $\hat{H}_{f_n}(x, y) = \hat{H}(x, y) + P(x) + Q(y)$  where  $P(x)$  and  $Q(y)$  are harmonic in  $B_1$  and bounded and continuous up to the boundary. Finally, for simply-connected domain  $\Omega$  let  $w(z)$  denote the conformal mapping of  $\Omega$  into  $B_1$ . Then one finds

$$(9.5) \quad H_n(x, y) = \hat{H}_{f_n}(w(x), w(y))$$

where  $\widehat{f_n}(x)$  is defined as  $\widehat{f_n}(w(z)) = f_n(z)/w(z)$ . We note again that in our case  $\widehat{f_n}(x) = \partial_\tau \theta + \frac{1}{n} \partial_\tau \varphi_\star$ .

From Lemma 9.2 and the construction of  $H_n$ , one can readily prove the following estimate on  $H_n$ .

**Lemma 9.3.** *The functions  $N_n(x, y)$  and  $H_n(x, y)$  satisfy the following estimates*

$$(9.6) \quad \begin{aligned} \left| H_n(x, y) - \log [\max\{|x - y|, \text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\}]^{-1} \right| &\leq C \\ N_n(x, y) &\leq C \\ H_n(x, x) &\leq C \end{aligned}$$

uniformly in  $n$ , where  $C$  depends on  $\Omega$  and  $\varphi_\star$ .

Using (9.6) we prove the following lemma that provides a lower bound on the intervortex distance as a function of the renormalized energy.

**Lemma 9.4.** *Let  $W(a)$  be the renormalized energy for the Dirichlet case. Then*

$$(9.7) \quad \rho_a \geq e^{-W(a) - Cn^2}$$

for some constant  $C$  that depends only on  $\Omega$  and  $\varphi_\star$ .

*Proof.* Since the domain is bounded we have from (9.6)

$$\begin{aligned} W(a) &= - \sum_{j \neq k} N_n(a_j, a_k) - \sum_j H_n(a_j, a_j) \\ &\geq \log[\min_{j \neq k} \{|a_j - a_k|\}]^{-1} + \log[\min\{\text{dist}(a_j, \partial\Omega)\}]^{-1} - Cn^2 \\ &\geq \log \rho_a^{-1} - Cn^2, \end{aligned}$$

where  $C$  depends on  $\varphi_\star$  and  $\Omega$ . Therefore,

$$\rho_a = e^{\log \rho_a} \geq e^{-W(a) - Cn^2}.$$

□

This lower bound is sufficient to continue solutions to asymptotically long times. In particular we finish with the

*Proof of Proposition 9.1.* Let  $a_j(t)$  be the solution to

$$\dot{a}_j = -\frac{1}{\pi} \nabla_{a_j} W(a)$$

in the Dirichlet case. Our initial data is chosen to ensure that  $\rho_\star = |\log |\log \varepsilon||^{-\frac{1}{6}}$ . From our assumptions on  $\rho_{a(0)}$  and  $n$  we have

$$W(a(0)) \leq C \frac{n^2}{\rho_{a(0)}} \leq C |\log |\log \varepsilon||^{\frac{5}{6}}.$$

Since  $W(a(t)) \leq W(a(0))$  then from Lemma 9.4 we see that

$$\begin{aligned} \rho_{a(t)}^{-1} &\leq C e^{W(a(t))+Cn^2} \leq C e^{W(a(0))+Cn^2} \\ &\leq C e^{C|\log|\log|\log\varepsilon|||^{\frac{5}{6}}} \leq C e^{\frac{1}{10}|\log|\log|\log\varepsilon|||} \\ &\leq |\log|\log\varepsilon||^{\frac{1}{6}} \end{aligned}$$

for all time; hence we can set  $\rho_\star = |\log|\log\varepsilon||^{-\frac{1}{6}}$ . Finally, this implies a time of existence up to

$$T = C|\log|\log\varepsilon||^{\frac{1}{2}} \frac{\rho_\star^2}{n} \geq C \frac{|\log|\log\varepsilon||^{\frac{1}{2}-\frac{1}{3}}}{|\log|\log|\log\varepsilon|||^{\frac{1}{4}}} \geq |\log|\log\varepsilon||^{\frac{1}{7}},$$

which certainly satisfies  $T \gtrsim \frac{1}{n}$ .  $\square$

## 10. HYDRODYNAMIC LIMIT

We start by examining the vortex density function  $\omega(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \delta_{a_j(t)} = \lim_{n \rightarrow \infty} \omega_n(t)$  which is the  $\mathcal{M}$ -limit of the cloud of delta functions given by the ODE (1.11). We follow the approach of Schochet [46] and Liu-Xin [36] for the planar Euler equations and Lin-Zhang [34] for the planar time dependent Ginzburg-Landau equation. Here we encounter some challenges due to the boundary condition, and this leads us to consider interior weak solutions.

In the following subsection we will relate this ODE limit to the energy density  $\frac{e_{\varepsilon_n}(u_{\varepsilon_n}(t))}{n\pi|\log\varepsilon_n|}$  of a corresponding PDE limit problem for an appropriately chosen sequence  $\varepsilon_n$  with appropriately constructed initial data.

**Proposition 10.1.** *Consider a sequence of initial data  $\{a_j(0)\}_{j=1}^n$ , and assume*

$$-\frac{1}{n^2} \sum_{j \neq k} N_n(a_j(0), a_k(0)) \lesssim 1$$

*for every  $n$ . Let  $a_j(t)$  solve (1.11) with  $W(a)$  in the Dirichlet case. If  $\omega_n = \frac{1}{n} \sum_{j=1}^n \delta_{a_j}$  then rescaling time  $\bar{t} = nt$ , we find  $\omega_n(t) \rightarrow \omega(t)$  in  $\mathcal{M}$  for all  $t$  and  $\omega$  is a generalized interior weak solution to*

$$\partial_{\bar{t}} \omega + \operatorname{div} (\nabla (\Delta_{\mathcal{N}}^{-1} \omega) \omega) = 0$$

*with  $\omega_0 = \lim \omega_n(0)$  defined by (1.18). Finally, we have  $v \in L_{loc}^2(\Omega)$  where  $v = \nabla \Delta_{\mathcal{N}}^{-1}(\omega)$ .*

We first show that the vortex density function  $\omega_n(t)$  satisfies an equation very close to (1.18). Following [3], let

$$\begin{aligned} \Delta G &= \sum_{j=1}^n 2\pi \delta_{a_j} \\ \partial_\nu G &= n \partial_\tau \theta + \partial_\tau \varphi_\star \end{aligned}$$

then  $G(x) = \sum_{j=1}^n N_n(x, a_j)$  where  $N_n(x, a_j)$  is defined by (9.2). If

$$S_n^j(x) = \sum_{k=1}^n N_n(x, a_k) - \log |x - a_j|$$

then from [3]

$$-\nabla_{a_j} W(a) = \nabla S_n^j(a_j)$$

where

$$\begin{aligned} \nabla S_n^j(x) &= \sum_{k \neq j}^n \nabla N_n(x, a_k) + \left[ \nabla N_n(x, a_j) - \frac{x - a_j}{|x - a_j|^2} \right] \\ &= \sum_{k \neq j}^n \nabla N_n(x, a_k) + \nabla H_n(x, a_j). \end{aligned}$$

Next note that  $\delta_{a_k} = \Delta N_n(\cdot, a_k)$ , so for a test function  $\chi \in C_0^\infty(\Omega)$  and with  $\omega_n = \frac{1}{n} \sum_{j=1}^n \delta_{a_j(t)}$  we have

$$\begin{aligned} \frac{1}{n} \partial_t \int \chi \omega_n(t) &= \frac{1}{n^2} \partial_t \sum_{j=1}^n \chi(a_j) = \frac{1}{n^2} \sum_{j=1}^n \partial_\ell \chi(a_j) [\dot{a}_j]_\ell \\ &= -\frac{1}{\pi} \frac{1}{n^2} \sum_{j=1}^n \partial_\ell \chi(a_j) [\nabla_{a_j} W(a)]_\ell = \frac{1}{\pi} \frac{1}{n^2} \sum_{j=1}^n \partial_\ell \chi(a_j) \partial_\ell S_n^j(a_j) \\ &= \frac{1}{\pi} \frac{1}{n^2} \sum_{j=1}^n \int \partial_\ell \chi(x) \partial_\ell S_n^j(x) \delta_{a_j}(x). \end{aligned}$$

Using the above identity for  $\nabla S_n^j$  and  $\delta_{a_j(x)}$  yields

$$\begin{aligned} \frac{1}{n} \partial_t \int \chi \omega_n(t) &= \frac{1}{\pi} \frac{1}{n^2} \sum_{j=1}^n \int \partial_\ell \chi(x) \left[ \sum_{k \neq j} \partial_\ell N_n(x, a_k) + \partial_\ell H_n(x, a_j) \right] \partial_m \partial_m N_n(x, a_j) dx \\ &= \frac{1}{\pi} \int \int_{y \neq z} \partial_\ell \chi(x) \partial_\ell N_n(x, y) \partial_m \partial_m N_n(x, z) \omega_n(y) \omega_n(z) dy dz dx \\ &\quad + \frac{1}{\pi} \frac{1}{n} \int \int \partial_\ell \chi(x) \partial_\ell H_n(x, y) \partial_m \partial_m N_n(x, y) \omega_n(y) dy dx \\ &= A_n + B_n. \end{aligned}$$

Following [36] we define the matrix-valued function  $\mathcal{K}(n, y, z; \eta)$

$$(10.1) \quad \mathcal{K}_{jk}(n, y, z, \eta) = \int_{\Omega} \eta(x) \partial_{x_j} N_n(x, y) \partial_{x_k} N_n(x, z) dx,$$

and after a short calculation, one can rewrite  $A_n$  and  $B_n$  as

$$\begin{aligned}
 (10.2) \quad A_n &= -\frac{1}{2\pi} \int \int_{y \neq z} (\mathcal{K}_{11} - \mathcal{K}_{22}) (n, y, z, (\partial_{x_1}^2 - \partial_{x_2}^2) \chi) \omega_n(y) \omega_n(z) dy dz dx \\
 &\quad - \frac{1}{\pi} \int \int_{y \neq z} \mathcal{K}_{12} (n, y, z, \partial_{x_1} \partial_{x_2} \chi) \omega_n(y) \omega_n(z) dy dz dx \\
 B_n &= \frac{1}{n\pi} \int \int \partial_m \partial_m (\partial_\ell \chi \partial_\ell H_n(x, y)) N_n(x, y) \omega_n(y) dy dx.
 \end{aligned}$$

We will show that as  $n \rightarrow \infty$ ,  $B_n$  converges to zero and  $A_n^j$ 's converge to the form of the generalized weak solution. However, in order to complete the proof, we prove two technical lemmas on the  $\mathcal{K}_{jk}$  and the vorticity maximal function (defined below).

The following lemma provides estimates on the Neumann functions  $N_n(x, y)$ . In particular we have:

**Lemma 10.2.** *The matrix functions  $\mathcal{K}_{jk}(n, y, z, \eta)$  defined in (10.1) satisfies the following estimates for  $y, z \in \Omega$  and  $\eta \in C_0^\infty(\Omega)$ :*

$$(10.3) \quad |(\mathcal{K}_{11} - \mathcal{K}_{22})(n, y, z, \eta)| \leq C$$

$$(10.4) \quad |\mathcal{K}_{12}(n, y, z, \eta)| \leq C$$

$$(10.5) \quad |\mathcal{K}_{11}(n, y, z, \eta)| + |\mathcal{K}_{22}(n, y, z, \eta)| \leq 2 \log |y - z| + C$$

where  $C$  depends only on  $\eta$ ,  $\varphi_*$ , and  $\Omega$ . Finally, we have the bound

$$(10.6) \quad \left| \nabla_x^k H_n(x, y) \right| \leq \frac{C}{\text{dist}(y, \partial\Omega)^k}$$

where  $C$  depends on  $k$ ,  $\varphi_*$ , and  $\Omega$ .

*Proof.* These estimates are similar to ones found in Delort [9] and Evans-Müller [16] for the associated Greens function on  $\mathbb{R}^2$ ; therefore, we sketch the proof of (10.4) following the argument of [16]. The proofs of (10.3) and (10.5) can be established by similar adjustments of arguments in [16].

To prove (10.4) one needs to examine the behavior of the gradient of  $H_n(x, p) = N_n(x, p) - \log |x - p|$  defined via (9.4) and (9.5). Since the test function  $\eta$  has compact support away from the boundary, it follows that  $\partial_{x_j} H_n(x, \cdot)$  is bounded for all  $x$  on the support of  $\eta$ , as in the proof of Lemma 9.2.

We can now write

$$\begin{aligned}
 |\mathcal{K}_{12}| &= \left| \int \eta(x) \left[ \frac{(x-y)_1}{|x-y|^2} + \partial_{x_1} H_n(x, y) \right] \left[ \frac{(x-z)_2}{|x-z|^2} + \partial_{x_2} H_n(x, z) \right] dx \right| \\
 &\leq \left| \int \eta(x) \left[ \frac{(x-y)_1}{|x-y|^2} \frac{(x-z)_2}{|x-z|^2} \right] \right| + \left| \int \eta(x) \left[ \frac{(x-y)_1}{|x-y|^2} \partial_{x_2} H_n(x, z) \right] \right| \\
 &\quad + \left| \int \eta(x) \left[ \partial_{x_1} H_n(x, z) \frac{(x-y)_2}{|x-y|^2} \right] \right| + \left| \int \eta(x) [\partial_{x_1} H_n(x, z) \partial_{x_2} H_n(x, z)] \right| \\
 &= I_1 + I_2 + I_3 + I_4
 \end{aligned}$$

Using the support of  $\eta$  and the explicit estimates in the proof of Theorem 1.1 of [16], it follows that  $I_1 \leq C$ .  $I_4 \leq C$  due to the uniform bounds on  $\nabla_x H_n(x, \cdot)$  for  $x$  having compact support away from the boundary, where  $C$  depends on the distance of the

support to the boundary. Finally, we consider the the bound on  $I_2$  and  $I_3$ , which can be handled by identical bounds. Due to the uniform bound on  $\nabla_x H_n(x, \cdot)$  away from the boundary, we have

$$\begin{aligned} I_2 &= \left| \int \eta(x) \left[ \frac{(x-y)_1}{|x-y|^2} \partial_{x_2} H_n(x, z) \right] \right| \lesssim \int_{\text{supp}(\eta)} \frac{1}{|x-y|} dx \\ &\lesssim \int_0^{\text{diam}(\Omega)} dr \lesssim 1. \end{aligned}$$

Combining the estimates yields (10.4).  $\square$

Define for any Radon measure  $\mu$  the maximal vorticity function  $M_r(\mu)$  of DiPerna-Majda [10]

$$M_r(\mu) = \sup_{x \in \Omega, 0 < t \leq T} \int_{B_r(x) \cap \Omega} |\mu(y, t)| dy$$

for  $0 < r \leq \frac{1}{2}$ . As in [37, 36, 34] we prove a decay estimate on  $M_r(\omega_n)$  below in order to pass to the limit in the main term  $A_n$ . In particular we have:

**Lemma 10.3.** *Suppose  $\{a_j(t)\}_{j=1}^n$  arise from the hypotheses of Proposition 10.1, then we can bound*

$$M_r(\omega_n(t)) \lesssim \frac{1}{\sqrt{|\log r|}} + \frac{1}{\sqrt{n}}$$

for all  $n$  and all  $r \leq 1$ . Furthermore,

$$M_r(\omega) \lesssim \frac{1}{\sqrt{|\log r|}}.$$

*Proof.* Following the structure of the argument in [36] we have for some positive integer  $k_x \leq n$ ,

$$\begin{aligned} |\log r| M_r^2(\omega_n(t)) &= |\log r| \left[ \frac{1}{n} \# \{a_j(t) \in B_r(x) \cap \Omega\} \right]^2 \\ &= |\log r| \frac{k_x(k_x - 1)}{n^2} + |\log r| \frac{k_x}{n^2} \\ &\lesssim \left[ \frac{1}{n^2} \sum_{|a_j - a_k| \leq r} [-N_n(a_j, a_k) + C] \right] + \frac{|\log r|}{n} M_r(\omega_n) \\ &\lesssim 1 + \frac{|\log r|}{n} M_r(\omega_n) \end{aligned}$$

where we used Lemma 9.2, Lemma 9.3, and  $\sum_{j \neq k} 1 \leq n^2$ . Since  $M_r(\omega_n) \leq 2\pi$  the bound follows.

For the bound on  $M_r(\omega)$  we have for  $\chi \in C^\infty$  where  $\chi = 1$  on  $B_r(x)$  and  $\chi = 0$  on  $\mathbb{R}^2 \setminus B_{2r}(x)$ ,  $x$  is chosen where  $\int_{B_r(x) \cap \Omega} \omega(t)$  is maximal, then

$$M_r(\omega) \leq \int \chi \omega = \lim_{n \rightarrow \infty} \int \chi \omega_n \leq \lim_{n \rightarrow \infty} M_{2r}(\omega_n) \lesssim \frac{1}{\sqrt{|\log r|}}.$$

$\square$

*Proof of Proposition 10.1.* We now examine the convergence behavior of  $A_n^j$  and  $B_n$ . From Lemma 10.2 one can follow the arguments of [46, 36] to establish the convergence of  $A_n^j$ . Looking at  $A_n^1$  and taking  $\chi \in C_0^\infty(\Omega)$  and setting  $\eta = (\partial_{x_1}^2 - \partial_{x_2}^2) \chi$ , we have

$$\begin{aligned} A_n^1 &= -\frac{1}{2\pi} \int \int_{\{|y-z| \geq r\} \cap \Omega} (\mathcal{K}_{11} - \mathcal{K}_{22})(n, y, z, \eta) \omega_n(y) \omega_n(z) dy dz dx \\ &\quad - \frac{1}{2\pi} \int \int_{\{0 < |y-z| < r\} \cap \Omega} (\mathcal{K}_{11} - \mathcal{K}_{22})(n, y, z, \eta) \omega_n(y) \omega_n(z) dy dz dx. \end{aligned}$$

Since  $(\mathcal{K}_{11} - \mathcal{K}_{22})(n, y, z, \eta)$  is continuous in each variable and bounded in the region then that term converges to

$$-\frac{1}{2\pi} \int \int_{\{|y-z| \geq r\} \cap \Omega} (\mathcal{K}_{11} - \mathcal{K}_{22})(\infty, y, z, \eta) \omega(y) \omega(z) dy dz dx.$$

On the other hand in the second region we have

$$\begin{aligned} &\left| \frac{1}{2\pi} \int \int_{\{0 < |y-z| < r\} \cap \Omega} (\mathcal{K}_{11} - \mathcal{K}_{22})(n, y, z, \eta) \omega(y) \omega(z) dy dz dx \right| \\ &\leq C \int_{\{0 < |y-z| < r\} \cap \Omega} \omega_n(y) \omega_n(z) dy dz \\ &\lesssim \|\omega_n\|_{L^1(\Omega)} \int_{\{|z| < r\} \cap \Omega} \omega_n(z) dz \\ &\lesssim M_r(\omega_n) \end{aligned}$$

and by Lemma 10.3 the term goes to zero as  $n \rightarrow \infty$  and  $r \rightarrow 0$ . This implies  $A_n^1 \rightarrow A^1$ . The convergence of  $A_n^2$  is much easier since the kernel is continuous on the entire domain.

Next, we show that  $B_n \rightarrow 0$ , and here we crucially use the compact support of the our test function  $\chi$ .  $B_n$  consists of three terms, depending on where the derivatives hit. We consider the worst case in which all derivatives hit  $H_n$ . Using (10.6) we get

$$\begin{aligned} &\frac{1}{n\pi} \left| \int \int \partial_\ell \chi \partial_m \partial_m \partial_\ell H_n(x, y) N_n(x, y) \omega_n(y) dy dx \right| \\ &\lesssim \frac{1}{n} \|\omega_n\|_{L^1} \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . The rest of the terms of  $B_n$  are estimated in a similar fashion.

Finally, we can prove the estimate on the kinetic energy in the fashion of Liu-Xin [36]. As in [36] one can use the decay of  $M_r(\omega) \rightarrow 0$  to prove that

$$(10.7) \quad - \int_{\{|y-z| \leq r\} \cap \Omega} \log |y-z| \omega(y) \omega(z) dy dz \lesssim 1.$$



Then for  $K$  a compact set in  $\Omega$  take a nonnegative test function  $\chi \in C_0^\infty(\Omega)$  with  $\chi = 1$  on  $K$ . Then

$$\begin{aligned} \int_K v^2 &\leq \int \chi v^2 = \int \int (\mathcal{K}_{11} + \mathcal{K}_{22}) \omega(y) \omega(z) dy dz \\ &= \int_{\{|y-z| < r\} \cap \Omega} (\mathcal{K}_{11} + \mathcal{K}_{22}) \omega(y) \omega(z) dy dz \\ &\quad + \int_{\{|y-z| \geq r\} \cap \Omega} (\mathcal{K}_{11} + \mathcal{K}_{22}) \omega(y) \omega(z) dy dz \\ &= A + B. \end{aligned}$$

Since  $B$  is away from the singularity, then we see immediately that  $B$  is bounded. The bound on  $A$  follows from (10.5) and (10.7).  $\square$

We are now in position to establish the hydrodynamic limit. The primary task is to approximate the initial data in a suitable way by quantized vortices that satisfy a good energy bound. Then we can use Proposition 10.1.

*Proof of Theorem 1.3.* We first approximate initial data for  $0 \leq \omega_0 \in \mathcal{M} \cap \dot{H}^{-1}(\Omega)$  in a suitable way so that we can use both Theorem 1.1 and Proposition 10.1.

1. Assume  $\text{supp } \omega_0 \subset \tilde{\Omega}$  with  $\text{dist}(\tilde{\Omega}, \partial\Omega) \geq C > 0$ . We then cover our set  $\Omega$  with nonoverlapping squares  $\{Q_j\}$ , where

$$Q_j \equiv j' \text{th square of side-length } h,$$

so there exist  $O(h^{-2})$  squares  $Q_j$  that cover  $\Omega$ . We then set

$$(10.8) \quad h = n^{-\frac{1}{4}}$$

so  $h^{-2} \ll n$ . We now define

$$\omega_0^n = \sum_{Q_j} \omega_{0,j}^n$$

where the  $\omega_{0,j}^n$  are set below. Next, set

$$\tilde{n}_j^h = \frac{n}{2\pi} \int_{Q_j} \omega_0$$

and  $n_j = \lfloor \tilde{n}_j^h \rfloor$ , then  $|n_j - \tilde{n}_j^h| < 1$  and

$$(10.9) \quad \left| \sum_j n_j - n \right| \lesssim h^{-2} = n^{\frac{1}{2}}.$$

Since  $\omega_0$  has compact support, then for all  $h \leq h_0 = h_0(\Omega)$  small enough, if  $Q_k \cap \partial\Omega \neq \emptyset$  then  $n_j = 0$ . If we set  $\hat{n} = \sum_j n_j$  then

$$n - Cn^{\frac{1}{2}} \leq \hat{n} \leq n$$

so  $\hat{n} \rightarrow \infty$  in the same rate as  $n \rightarrow \infty$ . We can then use  $\hat{n}$  instead of  $n$  in the discussion below; however, we relabel  $\hat{n}$  as  $n$  for simplicity.

Next, we slice  $Q_j$  into  $n_j$  thin rectangles of equal width. They will be aligned vertically and horizontally in alternating sequence, see Figure 2. In the center of

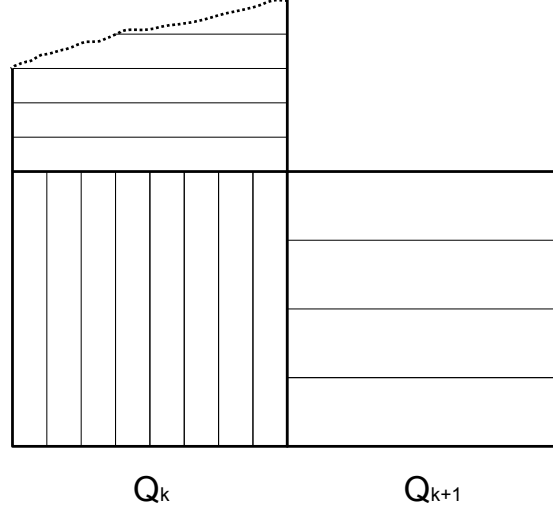


FIGURE 2. Construction of the rectangles in the  $Q_k$ 's

each of these subrectangles we label points  $\{a_{0,j}^1, \dots, a_{0,j}^{n_j}\}$ , so the distance between neighboring points is  $\frac{h}{n_j}$ . Finally, we let

$$(10.10) \quad \omega_{0,j}^n = \frac{1}{n} \sum_{k=1}^{n_j} \delta_{a_{0,j}^k}$$

where the  $a_{0,j}^k$  are defined above. In the worst-case scenario all vortices are located in a single cell with intervortex distance  $O(\frac{h}{n}) \approx n^{-\frac{5}{4}}$ , and we will need to check that this conforms to the correct bound on  $\rho_{a(0)}$ .

We claim that  $\omega_0^n \rightarrow \omega_0$  in  $\mathcal{M}(\Omega)$ . Let  $f_U$  denote the average of  $f$  on  $U$ . Then for  $\chi \in C_0^0(\Omega)$ ,  $|\int_{\Omega} \chi (\omega_0^n - \omega_0)| \leq \sum_{Q_j} \left| \int_{Q_j} (\chi - \chi_{Q_j})(\omega_0^n - \omega_0) \right| + \sum_{Q_j} |\chi_{Q_j}| \left| \int_{Q_j} \omega_0^n - \omega_0 \right| \rightarrow 0$  as  $n \rightarrow \infty$  from (10.8), (10.9), and the continuity of  $\chi$ . Therefore,  $\omega_0^n \rightarrow \omega_0$  in  $\mathcal{M}(\Omega)$ .

2. Finally, we claim that

$$(10.11) \quad -\frac{1}{n^2} \sum_{a_{0,i}^j \neq a_{0,k}^\ell} N_n(a_{0,i}^j, a_{0,k}^\ell) \lesssim 1.$$

Since the support of  $\omega_0$  lies in a compact set away from the boundary, then

$$\min\{\text{dist}(a_{0,j}^k, \partial\Omega)\} \geq C > 0$$

uniformly in  $n$ . Hence, we have  $|H_n(a_{0,j}^k, a_{0,i}^\ell)| \leq C$  uniformly in  $n$ . In particular, to establish (10.11) it is sufficient to prove

$$(10.12) \quad -\frac{1}{n^2} \sum_{a_{0,i}^j \neq a_{0,k}^\ell} \log |a_{0,i}^j - a_{0,k}^\ell| \lesssim 1.$$

We subdivide the sum into those vortex interactions arising from the same  $Q_j$ 's and those that arise from differing  $Q_k$ 's,

$$\begin{aligned} -\frac{1}{n^2} \sum_{a_{0,i}^j \neq a_{0,k}^\ell} \log |a_{0,i}^j - a_{0,k}^\ell| &= -\frac{1}{n^2} \sum_j \sum_{k \neq \ell} \log |a_{0,j}^k - a_{0,j}^\ell| - \frac{1}{n^2} \sum_{j \neq k} \sum_{i,\ell} \log |a_{0,j}^i - a_{0,k}^\ell| \\ &= A + B \end{aligned}$$

We now consider the sum  $A$ . Concentrating on a single  $Q_j$ , assume without loss of generality that the subrectangles are vertical and  $a_{0,j}^1$  is located at the origin. Then the vortices in this square are located along the  $x$ -axis with  $x$  values at  $\{0, \Delta, 2\Delta, \dots, (n_j - 1)\Delta\}$ , where  $\Delta = \frac{h}{n_j}$ . Summing over the log interactions yields

$$\begin{aligned} -\sum_{k \neq \ell} \log |a_{0,j}^k - a_{0,j}^\ell| &= -[(n_j - 1) \log |\Delta| + (n_j - 2) \log |2\Delta| + \dots + \log |(n_j - 1)\Delta|] \\ &\leq \frac{n_j(n_j - 1)}{2} \log \Delta^{-1} \leq \frac{n_j^2}{2} \log h^{-1} + \frac{n_j^2}{2} \log n \\ &\leq 3n_j^2 \log h^{-1}, \end{aligned}$$

since  $\log h^{-1} = \frac{1}{4} \log n$ . Now summing over the  $j$ 's yields and using that  $\frac{n_j}{n} \leq \frac{1}{2\pi} \int_{Q_j} \omega_0$ , we get

$$\begin{aligned} A &= -\frac{1}{n^2} \sum_j \sum_{k \neq \ell} \log |a_{0,j}^k - a_{0,j}^\ell| \lesssim \frac{1}{n^2} \sum_j n_j^2 \log h^{-1} \\ &\lesssim \sum_j \log h^{-1} \int_{Q_j} \omega_0(y) \int_{Q_j} \omega_0(z) \\ &\lesssim -\sum_j \int_{Q_j} \int_{Q_j} \omega_0(y) \log |y - z| \omega_0(z) dy dz \end{aligned}$$

Next we bound  $B$ . Let  $p_j$  denote the center of the square  $Q_j$ . Due to the alternating alignment of the subrectangles in Figure 2, we see that  $\left| \log |a_{0,j}^i - a_{0,k}^\ell| - \log |p_j - p_k| \right| \leq$

$C$ , even for neighboring squares. Therefore,

$$\begin{aligned}
B &= -\frac{1}{n^2} \sum_{j \neq k} \sum_{i, \ell} \log |a_{0,j}^i - a_{0,k}^\ell| \\
&= -\frac{1}{n^2} \sum_{j \neq k} \sum_{i=1}^{n_j} \sum_{\ell=1}^{n_k} \log |a_{0,j}^i - a_{0,k}^\ell| \\
&\leq C - \frac{C}{n^2} \sum_{j \neq k} \sum_{i=1}^{n_j} \sum_{\ell=1}^{n_k} \log |p_i - p_k| = C - \frac{C}{n^2} \sum_{j \neq k} n_j n_k \log |p_i - p_k| \\
&\lesssim 1 - \sum_{j \neq k} \log |p_i - p_k| \int_{Q_j} \omega_0(y) \int_{Q_k} \omega_0(z) \\
&\lesssim 1 - \sum_{j \neq k} \int_{Q_j} \int_{Q_k} \omega_0(y) \log |y - z| \omega_0(z) dy dz
\end{aligned}$$

Combining  $A$  and  $B$  together we find

$$\begin{aligned}
A + B &\lesssim 1 - \int_{\Omega} \int_{\Omega} \omega_0(y) \log |y - z| \omega_0(z) dy dz \\
&\lesssim 1 + \|\omega_0\|_{\dot{H}^{-1}(\Omega)} \left\| \mu_{\text{supp}(\omega_0)}(y) \int \log |y - z| \omega_0(z) \right\|_{H^1(\Omega)} \\
&\lesssim 1 + \|\omega_0\|_{\dot{H}^{-1}}^2 \lesssim 1,
\end{aligned}$$

where  $\mu_Q$  is the characteristic function on  $Q$ .

3. Now we complete the proof of the hydrodynamic limit. Set  $\varepsilon_n$  such that  $n = |\log |\log |\log \varepsilon_n|||^{-\frac{1}{4}}$  and

$$\omega_n(t) = \frac{1}{n} \frac{e_{\varepsilon_n}(u_{\varepsilon_n}(t))}{\pi |\log \varepsilon_n|}.$$

Given the initial measure  $\omega_0$ , we build our initial data  $u_{\varepsilon_n}(0)$  with vortices at  $\{a_{0,j}^k\}$  as generated above. The intervortex distance is no worse than

$$\rho_{a(0)} \geq C \frac{h}{n} \geq C n^{-\frac{3}{2}} \geq C |\log |\log |\log \varepsilon_n|||^{-\frac{3}{8}} \geq |\log |\log |\log \varepsilon_n|||^{-\frac{1}{3}},$$

which satisfies the requirements of Proposition 9.1.

We now take a test function  $\chi \in C_0^\infty(\Omega)$  then setting

$$\Delta(t) = \frac{1}{n^2} \int \chi \left( \frac{e_{\varepsilon_n}(u_{\varepsilon_n}(t))}{\pi |\log \varepsilon_n|} - \sum_{j=1}^n \delta_{a_j(t)} \right) dx$$

we have

$$\begin{aligned}
\frac{1}{n} \int_s^t \int_{\Omega} \chi \partial_t \omega_n &= \frac{1}{n} \int_{\Omega} \omega_n(t) \chi dx - \int_{\Omega} \omega_n(s) \chi dx \\
&= \frac{1}{n^2} \int_{\Omega} \chi(x) \left( \sum_{j=1}^n \delta_{a_j(t)} - \delta_{a_j(s)} \right) dx \\
&\quad + \Delta(t) - \Delta(s) \\
&= \frac{1}{n^2} \sum_{j=1}^n \chi(a_j(t)) - \chi(a_j(s)) + \Delta(t) - \Delta(s) \\
&= \frac{1}{n^2} \int_s^t \sum_{j=1}^n \nabla \chi(a_j(r)) \cdot \dot{a}_j(r) + \Delta(t) - \Delta(s) \\
&= -\frac{1}{\pi} \frac{1}{n} \int_s^t \nabla \chi(a_j(r)) \cdot \nabla_{a_j} W(a) + \Delta(t) - \Delta(s) \\
&\rightarrow -\frac{1}{4\pi} \int_s^t \int \int \int ((\partial_1^2 - \partial_2^2) \chi) (v_1^2 - v_2^2) dx dy dz dr \\
&\quad - \frac{1}{2\pi} \int_s^t \int \int \int \partial_1 \partial_2 \chi v_1 v_2 dx dy dz dr
\end{aligned}$$

as  $n \rightarrow \infty$ , using Proposition 10.1. Here we used the simple estimate

$$\frac{1}{n^2} \|\nabla \chi\|_{L^\infty} \left\| \frac{e_{\varepsilon_n}(u_{\varepsilon_n})}{\pi |\log \varepsilon_n|} - \pi \sum_{j=1}^n \delta_{a_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \lesssim \frac{1}{n^2} \rightarrow 0.$$

Since this is true for every positive  $s < t$ , we complete the proof of Theorem 1.3.  $\square$

## REFERENCES

- [1] AMBROSIO, L., SERFATY, S. A gradient flow approach to an evolution problem arising in superconductivity. *Comm. Pure Appl. Math.* **61**, 11 (2008), 1495–1539.
- [2] AMBROSIO, L., MAININI, E., SERFATY, S. Gradient flow of the Chapman-Rubinstein-Schatzman model for signed vortices. To appear in *Ann. IHP, Analyse nonlinéaire*.
- [3] BETHUEL, F., BREZIS, H., HÉLEIN, F.: *Ginzburg-Landau Vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser, Boston, 1994
- [4] BETHUEL, F., ORLANDI, G., SMETS, D.: Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics. *Duke Math. J.* **130**, 523–614 (2005)
- [5] BREZIS, H., CORON, J.-M., LIEB, E.: Harmonic maps with defects. *Comm. Math. Phys.* **107**, 649–705 (1986)
- [6] CHAPMAN, S. J. *A Hierarchy of Models for Type-II Superconductors* SIAM Review **42** (2000), 555–598.
- [7] CHAPMAN, S. J., RUBINSTEIN, J., SCHATZMAN, M. *A mean-field model of superconducting vortices*. *European J. Appl. Math.* **7** (1996), 97–111.
- [8] COLLIANDER, J.E., JERRARD, R.L.: Ginzburg-Landau vortices: weak stability and Schrödinger equation dynamics. *J. Anal. Math.* **77**, 129–205 (1999)
- [9] DELORT, J.-M., Existence de nappes de tourbillon en dimension deux, *J. Amer. Math. Soc.*, **4** (1991), 553–586.

- [10] DiPERNA, R. J.; MAJDA, A. J., *Concentrations in regularizations for 2-D incompressible flow*, Comm. Pure Appl. Math. **40**, (1987), 301–345.
- [11] DORSEY, A. T. Vortex motion and the Hall effect in type-II superconductors: A time-dependent Ginzburg-Landau theory approach. *Phys. Rev. B* **46**, 13 (Oct 1992), 8376–8392.
- [12] E, W.: Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity. *Phys. D* **77**, 383–404 (1994)
- [13] E, W. *Dynamics of vortex liquids in Ginzburg-Landau theories with applications to superconductivity*, Phys. Rev. B **50**, (1994), 1126–1135.
- [14] ESSMANN, U.; TRÄUBLE, H. The flux-line arrangement in the "intermediate state" of type II superconductors" *Physics Letters* **27A** (1968), 156–157.
- [15] EVANS, L. C. *Partial differential equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
- [16] EVANS, L. C.; MÜLLER, S. *Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity*. J. Amer. Math. Soc. **7** (1994), no. 1, 199–219.
- [17] LOPES FILHO, M. C., NUSSENZVEIG LOPES, H. J., XIN, Z. *Existence of vortex sheets with reflection symmetry in two space dimensions*. Arch. Ration. Mech. Anal. **158** (2001), no. 3, 235–257.
- [18] JERRARD, R.L.: Vortex dynamics for the Ginzburg–Landau wave equation. *Calc. Var. Partial Differential Equations* **9**, 683–688 (1999)
- [19] JERRARD, R. L. *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), 721–746.
- [20] JERRARD, R.L., SONER, H.M.: The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations* **14**, 151–191 (2002)
- [21] JERRARD, R.L., SONER, H.M.: Dynamics of Ginzburg–Landau vortices. *Arch. Rational Mech. Anal.* **142**, 99–125 (1998)
- [22] JERRARD, R.L., SPIRN, D.: Refined Jacobian estimates for Ginzburg-Landau functionals. *Indiana Univ. Math. Jour.* **56**, 135–186 (2007)
- [23] JERRARD, R.L., SPIRN, D.: Refined Jacobian estimates and Gross-Pitaevsky vortex dynamics *Arch. Ration. Mech. Anal.* (2008)
- [24] KOPNIN, N. B., IVLEV, B. I., AND KALATSKY, V. A. The flux-flow hall effect in type ii superconductors. an explanation of the sign reversal. *Journal of Low Temperature Physics* **90**, 1 (01 1993), 1–13.
- [25] KURZKE, M., MELCHER, C., MOSER, R.: Vortex motion for the Landau-Lifshitz-Gilbert equation with spin transfer torque. *SIAM J. Math. Anal.* **43**, 1099–1121.
- [26] KURZKE, M., MELCHER, C., MOSER, R., AND SPIRN, D. Dynamics for Ginzburg-Landau vortices under a mixed flow. *Indiana Univ. Math. J.* **58**, 6 (2009), 2597–2621.
- [27] KURZKE, M., SPIRN, D.  $\Gamma$ -stability and vortex motion in type II superconductors. *Communications in Partial Differential Equations* **36**, 2 (2011), 256 – 292.
- [28] KURZKE, M., SPIRN, D.: Quantitative equipartition of the Ginzburg-Landau energy with applications. *Indiana U. J. Math.*, to appear.
- [29] KURZKE, M., SPIRN, D.: Vortex liquids and the superconducting Hall effect. *in preparation*.
- [30] LIN, F.-H.: Some dynamical properties of Ginzburg-Landau vortices. *Comm. Pure Appl. Math.* **49**, 323–359 (1996)
- [31] LIN, F.-H.: Vortex dynamics for the nonlinear wave equation. *Comm. Pure Appl. Math.* **52**, 737–761 (1999)
- [32] LIN, F.H.; LIN, T.C. Minimax solutions of the Ginzburg-Landau equations. *Selecta Math.* **3** (1997), 99–113.
- [33] LIN, F.-H., XIN, J.: On the Incompressible Fluid Limit and the Vortex Motion Law of the Nonlinear Schrödinger Equation. *Comm. Math. Phys.* **200**, 249–274 (1999)
- [34] LIN, F. H.; ZHANG, P. *On the hydrodynamic limit of Ginzburg-Landau vortices*. Discrete Contin. Dynamic Systems **6** (2000), no. 1, 121–142.
- [35] LIN, F.; ZHANG, P. On the hydrodynamic limit of Ginzburg-Landau wave vortices. *Comm. Pure Appl. Math.* **55**, 7 (2002), 831–856.

- [36] LIU, J.G.; XIN, Z. Convergence of the point vortex method for 2-D vortex sheet. *Math. Comp.* **70** (2001), 595–606.
- [37] LIU, J.G.; XIN, Z. Convergence of Vortex Methods for Weak Solutions to the 2-D Euler Equations with Vortex Sheet Data. *Comm. Pure Appl. Math.* **48** (1995), 611–628.
- [38] MAJDA, A.J. Remarks on weak solutions for vortex sheets with a distinguished sign. *Indiana Univ. Math. J.* **42** (1993), 921–939.
- [39] MIOT, E. Dynamics of vortices for the complex Ginzburg-Landau equation. *Anal. PDE* **2**, 2 (2009), 159–186.
- [40] PERES, L.; RUBINSTEIN, J., Vortex dynamics in  $U(1)$  Ginzburg-Landau models, *Phys. D*, **64** (1993), 299–309.
- [41] ROITBERG, Y.: *Elliptic Boundary Value Problems in the Space of Distributions*. Mathematics and its Applications, 498. Kluwer Academic Publications, Dordrecht, 1999
- [42] SANDIER, E. Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.* **152** (1998), 379–403.
- [43] SANDIER, E., SERFATY, S.: Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.* **57**, 1627–1672 (2004)
- [44] SANDIER, E., SERFATY, S. *Product Estimates for Ginzburg-Landau and corollaries*. *J. Funct. Anal.* **211** (2004), no. 1, 219–244.
- [45] SANDIER, E., SORET, M.  $\mathbb{S}^1$ -Valued Harmonic Maps with High Topological Degree: Asymptotic Behavior of the Singular Set. *Potential Anal.* **13** (2000), 169–184.
- [46] SCHOCHET, S., The point vortex method for periodic weak solutions of the 2D Euler equations, *Comm. Pure Appl. Math.*, **49** (1996) 911–965.
- [47] SERFATY, S. *Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow, part I: Study of the perturbed Ginzburg-Landau equation*, *Journal Eur. Math Society* , 9, No 2, (2007), 177–217.
- [48] SERFATY, S. *Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow. II. The dynamics*. *J. Eur. Math. Soc. (JEMS)* **9** (2007), no. 3, 383–426.
- [49] SERFATY, S. *Gamma-convergence of gradient flows on Hilbert and metric spaces and applications*, to appear in *Comm. Pure Appl. Analysis*.
- [50] SERFATY, S., TICE, I. *Ginzburg-Landau vortex dynamics with pinning and strong applied currents*. To appear in *Arch. Rat. Mech. Anal.*
- [51] SPIRN, D. *Vortex dynamics of the full time-dependent Ginzburg-Landau model*. *Comm. Pure Appl. Math.* **55** (2002), 537–581.

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